PHYS 306: HOMEWORK ASSIGNMENT No. 4: ORBITS IN GRAVITATIONAL FIELDS

(Feb 7th, 2017)

HOMEWORK DUE: FRIDAY, FEB 17th, 2017

To be handed in during class- Late Homework will not be accepted

QUESTION (1): We consider some of the different orbits around a central potential $V(r) = -V_o/r$.

1(a) Consider first the elliptic orbit, depicted in Fig. 1, whose trigonometric form is $r(\phi) = r_o/(1 + e\cos\phi)$, with eccentricity e such that $0 < e < 1$. Show that this form can be rewritten in Cartesian coordinates as

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} \;=\; 1 \tag{0.1}$$

where the semimajor and semi-minor axes a and b are given in terms of $r_o = L^2/mV_o$, the "latus rectum", by $a = r_o/(1-e^2)$ and $b = r_o/(1-e^2)^{1/2}$ (and we note that r_{min}, the perigee distance, is $r_o/(1+e) = a(1-e)$).

1(b) Now show that when $e = 1$, we can write the curve as a parabola satisfying the equation

$$2r_o x + y^2 \;=\; r_o^2 \tag{0.2}$$

QUESTION (2) Now we are going to consider the radial equation for a non-rotating black hole - the remarkable thing is how much we can learn using only elementary analysis.

(a) In General Relativity it is found that the radial equation of an object orbiting a non-rotating black hole has the form

$$\dot{r}^2 \;+\; (1 - 2V_o/r)\left[\frac{l^2}{r^2} + 1\right] \;\;=\;\; E^2 \tag{0.3}$$

where r is the radial coordinate, l is the angular momentum, and E is the total energy; the potential $V_o = GM_o$, where M_o is the mass of the black hole. Show, using the standard substitution $u = 1/r$, and rewriting $\dot{r}$ in terms of $\dot{\phi}$ and $u' = du/d\phi$, that we can write a differential equation for $u(\phi)$ of form

$$u'' + u - \frac{V_o}{l^2} - 3V_o u^2 = 0 \tag{0.4}$$

(b) Now we employ a common manouevre in the theory of differential equations. Suppose that r is large, so that u is small. Then we treat the term $\propto u^2$ in the General Relativistic differential eqtn. in (0.4) for $u(\phi)$ as a *perturbation* on the differential equation obtained when this term is absent - for which the solution is the the known "Newtonian solution". Substitute this Newtonian solution for $u(\phi)$ into equation (0.4), and thereby write down a new form for this differential equation in terms of the eccentricity e and $\cos(\phi - \phi_o)$, where ϕ_o is the angle of periastron.

(c) We can drop the term $\propto \cos^2(\phi - \phi_o)$, and also the term equal to $-3V_o^3/l^4$, from the equation you derived in (b). Having done this, you will find a resulting simplified differential equation which looks like that of an oscillator subject to a periodic driving force $\propto \cos(\phi - \phi_o)$ (remember, we are looking at u as a function of ϕ). Find the solution to this differential equation.

END of 4th HOMEWORK ASSIGNMENT

Fig. 1