Home work 7.

Q1 (a)

The center of mass is at the center due to the symmetry. $I_{xx} = I_{yy} = I_{zz} = I$

$I_{xy} = I_{yz} = I_{zx} = 0$

$$I = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_a^b dr\, r^2 \sin\theta\, \rho\, (r\cos\theta)^2$$

$$= \frac{8}{15}\pi\rho(b^5 - a^5).$$

the total mass is

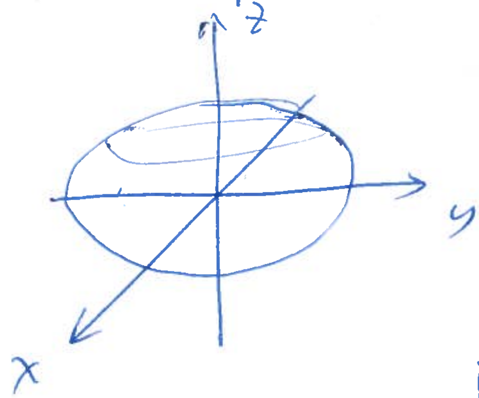
$$m = \rho V = \frac{4}{3}\pi\rho(b^3 - a^3)$$

$$\Rightarrow \rho = \frac{3}{4\pi} m \frac{1}{b^3 - a^3}$$

$$\Rightarrow I = \frac{2}{5} m \frac{b^5 - a^5}{b^3 - a^3}$$

3

(b) Ellipsoidal



we start from I_c,

(I_{a_0} and I_b can be calculate in the same way)

By definition

$$I_c = \iiint dx\,dy\,dz \quad \rho(x^2+y^2)$$

redefine $x' = \frac{x}{a}$, $y' = \frac{y}{b}$, $z' = \frac{z}{c}$.

then $I_c = \iiint dx' dy' dz' \; abc\,\rho\,(a^2 x'^2 + b^2 y'^2)$.

The surface is now spheric $x'^2 + y'^2 + z'^2 = 1$.

We change to spheric coordinate.

$$I_C = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^1 dr \cdot abc \cdot r^2 \sin\theta \; \rho \cdot$$

$$(a^2 r^2 \sin^2\theta \cos^2\varphi + b^2 r^2 \sin^2\theta \sin^2\varphi)$$

$$= \rho abc \cdot \frac{1}{5}\left(\frac{4}{3}\pi a^2 + \frac{4}{3}\pi b^2\right)$$

4.

$$= \frac{4}{15}\pi abc\,\rho(a^2+b^2).$$

the total mass is

$$m = \frac{4}{3}\pi abc \cdot \rho$$

so $I_c = \frac{1}{5} m (a^2 + b^2)$.

Similarly.

$$I_a = \frac{1}{5} m (b^2 + c^2)$$

$$I_b = \frac{1}{5} m (a^2 + c^2).$$

1(c) The ~~hollow~~ sphere of radius R_0 has moments of inertia

$$I_a' = I_b' = I_c' = \frac{8}{15}\pi \rho R_0^5$$

So the total moments of inertia is

3.

$$I_c = I_c^{ellip} - I_c' = \frac{4}{15}\pi \rho abc (a^2 + b^2)$$

$$- \frac{8}{15}\pi \rho R_0^5$$

total mass

$$m = \frac{4}{3}\pi \rho abc - \frac{4}{3}\pi \rho R_0^3$$

$\Rightarrow$

$$I_c = \frac{4}{15}\pi\rho\left(abc(a^2+b^2) - 2R_0^5\right)$$

$$= \frac{4}{15}\pi \cdot \frac{m}{\frac{4}{3}\pi(abc - R_0^3)}\left(abc(a^2+b^2) - 2R_0^5\right)$$

$$= \frac{m}{5} \cdot \frac{abc(a^2+b^2) - 2R_0^5}{abc - R_0^3}$$

Similarly

$$I_a = \frac{m}{5}\,\frac{abc(b^2+c^2) - 2R_0^5}{abc - R_0^3}$$

$$I_b = \frac{m}{5}\,\frac{abc(a^2+c^2) - 2R_0^5}{abc - R_0^3}$$

Q2 (a) The Euler's equation is

$$I_1 \dot{\Omega}_1 + (I_3 - I_2)\Omega_3 \Omega_2 = 0$$
$$I_2 \dot{\Omega}_2 + (I_1 - I_3)\Omega_3 \Omega_1 = 0$$
$$I_3 \dot{\Omega}_3 + (I_2 - I_1)\Omega_1 \Omega_2 = 0 .$$

write $\Omega_1(t) = w_1(t)$
$\Omega_2(t) = w_2(t)$
$\Omega_3(t) = \Omega_0(t) + w_3(t)$

and $w_{1,2,3} \ll \Omega_0$.

we have

$$I_1 \dot{w}_1 + (I_3 - I_2)(\Omega_0 + w_3) w_2 = 0$$
$$I_2 \dot{w}_2 + (I_1 - I_3)(\Omega_0 + w_3) w_1 = 0 .$$
$$I_3(\dot{\Omega}_0 + \dot{w}_3) + (I_2 - I_1) w_1 w_2 = 0$$

the lowest order $O(w_1)$ gives

$$I_1 \dot{w}_1 + (I_3 - I_2)\Omega_0 w_2 = 0$$
$$I_2 \dot{w}_2 + (I_1 - I_3)\Omega_0 w_1 = 0$$
$$I_3 \dot{\Omega}_0 = 0$$

So Ω_0 is a constant

when $t=0$. $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$

so $\Omega_0 = \Omega_3$.

$$I_1 \dot{w}_1 + (I_3 - I_2)\Omega_3 w_2 = 0$$
$$I_2 \dot{w}_2 + (I_1 - I_3)\Omega_3 w_1 = 0$$

assume the solution has the following oscillating form.

$$w_1 = A_1 \cos(\lambda t + \varphi_1)$$
$$w_2 = A_2 \cos(\lambda t + \varphi_2)$$

we have

$$-I_1 A_1 \lambda \sin(\lambda t + \varphi_1) + (I_3 - I_2)\Omega_3 A_2 \cos(\lambda t + \varphi_2) = 0$$
$$-I_2 A_2 \lambda \sin(\lambda t + \varphi_2) + (I_1 - I_3)\Omega_3 A_1 \cos(\lambda t + \varphi_1) = 0$$

to have both equation satisfied we need

$$\varphi_1 - \frac{\pi}{2} = \varphi_2$$

Then

$$-I_1 A_1 \lambda + (I_3 - I_2)\Omega_3 A_2 = 0$$

$$(I_1 - I_3)\Omega_3 A_1 + I_2 A_2 \lambda = 0.$$

So

$$\begin{pmatrix} -I_1\lambda & (I_3-I_2)\Omega_3 \\ (I_1-I_3)\Omega_3 & I_2\lambda \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

to have nontrivial solution, we have.

$$I_1 I_2 \lambda^2 + (I_3 - I_2)(I_1 - I_3)\Omega_3^2 = 0.$$

if $(I_3 - I_2)(I_1 - I_3) < 0$, we get the oscillatory solution. with frequency

$$\lambda = \sqrt{\frac{(I_3-I_2)(I_3-I_1)}{I_1 I_2}}\,\Omega_3$$

and $\dfrac{A_1}{A_2} = \dfrac{I_1\lambda}{(I_3-I_2)\Omega_3}$

So the solution can be written as

$$w_1 = A\cos(\lambda t + \varphi)$$

$$w_2 = \sqrt{\frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}}\, A\sin(\lambda t + \varphi)$$

since $t=0$, $w_2 = 0$, $\omega_1 = \overline{w_1}$

so we have

$$\varphi = 0, \qquad A = \overline{w_1}$$

3. So $w_1 = \overline{w_1}\cos\lambda t$

$$w_2 = \sqrt{\frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}}\,\overline{w_1}\sin\lambda t$$

2(C). for the second ~~order~~ nonzero order. we have

$$I_3 \dot{w}_3 + (I_2 - I_1)\, w_1 w_2 = 0.$$

We use the previous solution.

$$I_3 \dot{\omega}_3 + (I_2 - I_1)\,\overline{w_1}^2 \sqrt{\frac{(I_3 - I_2) I_2}{(I_3 - I_1) I_1}} \sin\lambda t \cos\lambda t$$

$$= 0$$

So $\dot{\omega}_3 = \frac{I_2 - I_1}{2I_3}\sqrt{\frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}}\,\bar{\omega}_1^2 \sin 2\lambda t$.

$$\omega_3 = \frac{\bar{\omega}_1^2}{\Omega_3}\frac{I_1 - I_2}{I_3 - I_1}\cdot\frac{I_2}{4I_3}\cos 2\lambda t + C.$$

The initial condition $\omega_3(t=0) = 0$.

So $C = \frac{I_2}{4I_3}\cdot\frac{I_2 - I_1}{I_3 - I_1}\frac{\bar{\omega}_1^2}{\Omega_3}$

$$\Omega_3(t) = \Omega_3\left(1 + \frac{I_2}{2I_3}\frac{I_2 - I_1}{I_3 - I_1}\frac{\bar{\omega}_1^2}{\Omega_3^2}\sin^2\lambda t\right)$$

(c) $\vec{L} = I\cdot\vec{\Omega}$

Since there is no external torque. we have $\vec{L}$ as a constant. So we only have to calculate the $\vec{L}$ at $t=0$.

$$I = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix} \qquad \vec{\Omega}(t=0) = \begin{pmatrix} \bar{\omega}_1 \\ 0 \\ \Omega_3 \end{pmatrix}$$

so $\vec{L} = (I_1 \bar{w}_1, \ 0, \ I_3 \Omega_3)$.

$$|\vec{L}| = \sqrt{I_1^2 \bar{w}_1^2 + I_3^2 \Omega_3^2}$$