HW 6, Question 1: Forces in a Rotating Frame

Consider a solid platform, like a "merry-go round", that rotates with respect to an inertial frame of reference. It is oriented in the xy plane such that the axis of rotation is in the z-direction, i.e. $\boldsymbol{\omega} = \omega\hat{z}$, which we can take to point out of the page. Now, let us also consider a person walking on the rotating platform, such that this person is walking radially outward on the platform with respect to their (non-inertial) reference frame with velocity $\mathbf{v} = v\hat{r}$.

In what follows for Question 1, directions in the non-inertial reference frame (i.e. the frame of the walking person) are defined in the following manner: the x'-direction points directly outward from the center of the merry-go round; the y' points in the tangential direction of rotation in the plane of the platform; and the z'-direction points out of the page, and in the same direction as the z-direction for the inertial reference frame.

(a) Identify all forces.

Since the person's reference frame is non-inertial, we need to account for the various fictitious forces given the description above, as well as any other external/reaction forces. The angular rate of rotation is constant, and so the Euler force $\mathbf{F}_{\text{eul}} = m(\dot{\boldsymbol{\omega}} \times \mathbf{r}) = \vec{0}$. The significant forces acting on the person are:

- the force due to gravity, $\mathbf{F}_{\text{g}} = \text{m}\mathbf{g} = mg(-\hat{z}')$,
- the (fictitious) centrifugal force, $\mathbf{F}_{\text{cen}} = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\omega^2 r\hat{x}'$,
- the (fictitious) Coriolis force, $\mathbf{F}_{\text{cor}} = 2m\mathbf{v} \times \boldsymbol{\omega} = 2m\omega v(-\hat{y}')$,
- the normal force the person experiences from the rotating platform $\mathbf{N} = N\hat{z}'$, and
- a friction force acting on the person, $\mathbf{f}$, in the $x'y'$ plane.

In order for the forces to balance, the friction force $\mathbf{f}$ must counteract all other forces acting on the system. In other words, $\mathbf{f}$ has components in the x'- and y'-directions: $\mathbf{f} = f_x(-\hat{x}') + f_y\hat{y}$. If we add up all components and set all sum-of-force components equal to zero, we get:

$$f_x = m\omega^2 r, \tag{1}$$
$$f_y = 2mv\omega, \tag{2}$$
$$N = mg. \tag{3}$$

(b) Find the point of slippage.

The person with velocity $\mathbf{v}$ will not slip so long as the friction force balances the combination of centrifugal and Coriolis forces, and the friction force does not exceed a value at which the fictitious forces dominate. However, slippage will occur when $\mathbf{f}$ reaches its maximum value: $f_{\text{max}} = \mu_s N$, where μ_s is the coefficient of static friction. We can use this limit and Equations 1-3 to find r as a function of all other known quantities:

$$\begin{aligned} f^2 &= f_x^2 + f_y^2 \\ &= (m\omega^2 r_{\text{slip}})^2 + (2mv\omega)^2 \\ &= \mu_s^2 (mg)^2 \implies \boxed{r_{\text{slip}} = \frac{1}{\omega^2}\sqrt{\mu_s^2 g^2 - 4v^2\omega^2}} \end{aligned} \tag{4}$$

(b) Find value for r_{slip}.

We're asked to consider a scenario where $\omega = 1$ rad s^{-1}, $v = 1$ m s^{-1}, and $\mu_s = 1/2$. We can take $g = 9.806$ m s^{-2}, the local value of acceleration due to gravity. Plugging all of these values into Equation 4, we find that

$$\boxed{r_{\text{slip}} \approx 4.48 \text{ m}} \tag{5}$$

HW 6, Question 2: Coriolis Force on a Moving Train

Let's consider a train in France at a latitude $\theta = 45$ degrees (i.e. in the northern hemisphere) that travels in the North direction at a constant speed v. As mentioned in the homework prompt, the center of mass is located at a height z_0 and the two rails are separated by a distance w_0.

(a) Find ratio of forces exerted by the rails onto the wheels.

We ultimately want to examine the forces experienced by the rails that the train travels on. In order to do so, we need to identify all forces acting on the rigid body:

- the force due to gravity, $\mathbf{F}_\mathrm{g} = \mathrm{m}\mathbf{g}$, acting at the center of mass of the train,
- the (fictitious) centrifugal force, $\mathbf{F}_\mathrm{cen} = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, acting at the center of mass of the train,
- the (fictitious) Coriolis force, $\mathbf{F}_\mathrm{cor} = 2m\mathbf{v} \times \boldsymbol{\omega}$, since the train is moving,
- the reaction force that the eastern rail exerts on the eastern wheel, $\mathbf{f}_1$, and
- the reaction force that the western rail exerts on the western wheel, $\mathbf{f}_2$.

In the case of Earth's rotation, the rotation rate $|\boldsymbol{\omega}| = \omega = 7.3 \times 10^{-5}$ rad/s is very small, so that terms that are proportional to ω^2 are negligible when compared to the Coriolis force, since the latter term is proportional to ω. We therefore only need to consider the significant forces acting on the system, which add to zero since the train keeps moving at a constant speed v:

$$\mathbf{F} \approx m\mathbf{g} + 2m\mathbf{v} \times \boldsymbol{\omega} + \mathbf{f}_1 + \mathbf{f}_2 = \mathbf{0} \tag{6}$$

We can take components of Equation 6 in the vertical and horizontal directions in the local, non-inertial frame of reference. The direction of the Coriolis force can be determined since the definition of the coordinate system dictates that $\boldsymbol{\omega}$ points in the direction of the axis of rotation, and that $\mathbf{v}$ points due North; the right hand rule requires that the vector $\mathbf{v} \times \boldsymbol{\omega}$ points due East. It has a magnitude of $|\mathbf{F}_\mathrm{cor}| = F_\mathrm{cor} = 2mv\omega \sin\theta$. Since we neglect the small contribution of $\mathbf{F}_\mathrm{cen}$, the local value of the gravitational acceleration $g = 9.806$ m s^{-2} and it points straight down towards the center of the Earth.

Figure **??** shows the various force vectors acting at their respective points in the rigid body of the train. With these directions, we can write the general y-component and z-component force equations,

$$\Sigma F_z = f_{1,z} + f_{2,z} - mg = 0 \tag{7}$$
$$\Sigma F_y = F_\mathrm{cor} - f_{1,y} - f_{2,y} = 0. \tag{8}$$

One initial approximation that we can make is that that the eastern rail experiences a much larger horizontal reaction force that opposes the Coriolis term than the western rail. From Equation 8, we therefore require that the sum $f_{1,x} + f_{2,x} \approx f_{1,x}$; by extension, this approximately means that $\mathbf{f}_2$ points strictly in the vertical direction, since it has no significant horizontal component and so $\mathbf{f}_2 \approx f_{2,y}\hat{z}$, while $\mathbf{f}_1$ has both horizontal and vertical components. If we make the substitution $|\mathbf{f}_2| \approx f_{2,y} = f_2$, we can re-write the component equations as

$$\Sigma F_z \approx f_{1,z} + f_2 - mg = 0 \tag{9}$$
$$\Sigma F_y \approx F_{\text{cor}} - f_{1,y} = 0. \tag{10}$$

Equations 9 and 10 contain three unknowns, so we cannot uniquely solve for each component using these two equations alone. However, we can proceed further with a few more approximations using two slightly different methods:

Method 1: Equilibrium of Forces and Torques

Since the train remains on the track and moves at a constant speed, the sum of all forces *and of all torques* must cancel to yield no net terms. We can add all moments of force and require that their sum be zero; in order to do so, as with calculations of angular momentum, we must select a reference point to measure moments from. In Figure **??**, the reference point is chosen (arbitrarily) to be at the western wheel (where $\mathbf{f}_2$ acts). The sum of all moments relative to this point are:

$$\begin{aligned}
\Sigma \mathbf{M} &= \cancel{(\mathbf{0} \times \mathbf{f}_2)} + (\mathbf{r}_{\text{COM}} \times \mathbf{F}_{\text{g}}) + (\mathbf{r}_{\text{COM}} \times \mathbf{F}_{\text{cor}}) + (\mathbf{r}_2 \times \mathbf{f}_1) \\
&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & w_0/2 & z_0 \\ 0 & 0 & -mg \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & w_0/2 & z_0 \\ 0 & F_{\text{cor}} & 0 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & w_0 & 0 \\ 0 & f_{1,y} & f_{1,z} \end{vmatrix} \\
&= \hat{x}\left(-\frac{mgw_0}{2} - z_0 F_{\text{cor}} + w_0 f_{1,z} \right) \qquad (11) \\
&= 0
\end{aligned}$$

Using Equation 11, we can find the value of $f_{1,z}$ in terms of the forces we can compute and in terms of the distances where the forces act:

$$f_{1,z} = \frac{mg}{2} + \frac{z_0}{w_0} F_{\text{cor}} \tag{12}$$

Method 2: Force Approximations

Instead of invoking torques, we can make further approximations of the force components through physical arguments. Let's the consider the simplest, "everyday" case, where we ignore the non-inertial forces and simply consider the force due to gravity and the two reaction forces on the rails. In the simplest case, the reaction forces are equal, and so $f_1 = f_2 = mg/2$ and each point in the $\hat{z}$ direction, so that the sum of the two forces perfectly balance the weight of the train.

In the case at hand, where we consider the Coriolis force to first order in ω, Equations 9 and 10 suggest that $\mathbf{f_1}$ and $\mathbf{f_2}$ are slightly modified. From Equation 10, we see that $f_{1,y} = f_1 \sin\phi = F_{\text{cor}}$, where $\tan\phi = F_{\text{cor}}/F_g$ (see Figure 1). Given the assumptions of how the reaction forces are generated, $f_{1,z}$ should be equal to $mg/2$ with the addition of a term that is proportional to $\mathbf{F}_{\text{cor}}$. From Equation 9, we see that the Coriolis term enters in the z-component Equation as $f_1 \cos\phi = (f_1 \sin\phi)\cot\phi = F_{\text{cor}} \cot\phi$. If we further assume that this component of the Coriolis term, as with the weight, is distributed evenly between the two rails, then we can write the first-order equation for f_2 as

$$f_2 = \frac{mg}{2} - \frac{1}{2}\cot\phi F_{\text{cor}}$$

since one would expect the eastward force to slightly lessen the reaction exerted by the western rail. For an order-of-magnitude calculation, we can require that $\tan\phi \sim z_0/(w_0/2) = 2z_0/w_0$. Therefore, with the above equation for f_2, we can find the modified equation for $f_{1,z}$ using Equation 9:

$$\begin{aligned} f_{1,z} &\approx \frac{1}{2}\left(mg + \cot\phi \mathbf{F}_{\text{cor}}\right) \\ &= \frac{mg}{2} + \frac{z_0}{w_0} F_{\text{cor}} \end{aligned} \tag{13}$$

Conclusions

The combination of Equations 9, 10 and 12 allows us to uniquely solve the system for the components we want. Since we're asked to find the ratio of the magnitudes of rail forces, we must first compute the magnitude of $\mathbf{f}_1$; we can do this by adding the components in quadrature and substituting the results from Equations 10 and 12:

$$\begin{aligned} f_1^2 &= f_{1,y}^2 + f_{1,z}^2 \\ &= F_{\text{cor}}^2 + \left(\frac{mg}{2} + \frac{z_0}{w_0} F_{\text{cor}}\right)^2 \\ &= F_{\text{cor}}^2\left(1 + \frac{z_0^2}{w_0^2}\right) + \frac{(mg)^2}{4} + \frac{mgz_0}{w_0} F_{\text{cor}} \\ &\approx \frac{(mg)^2}{4} + \frac{mgz_0}{w_0} F_{\text{cor}} \end{aligned} \tag{14}$$

where, in the final form of Equation 14, we note that $F_{\text{cor}}^2 \propto \omega^2$ and drop this term since ω is very small. In order to find the ratio f_1/f_2, we need to use Equation 9 and 12 in order to find f_2,

$$f_2 = mg - f_{1,z} = \frac{mg}{2} - \frac{z_0}{w_0} F_{\text{cor}} \tag{15}$$

and use the same approximation method we employed when deriving Equation 14 to find f_2^2. The square of the ratio of force magnitudes becomes

$$\begin{aligned}
\frac{f_1^2}{f_2^2} &= \frac{\frac{(mg)^2}{4} + \frac{mgz_0}{w_0} F_{\text{cor}}}{\frac{(mg)^2}{4} - \frac{mgz_0}{w_0} F_{\text{cor}}} \\
&= \left(1 + \frac{4z_0}{mgw_0} F_{\text{cor}}\right)\left(1 - \frac{4z_0}{mgw_0} F_{\text{cor}}\right)^{-1} \\
&\approx \left(1 + \frac{4z_0}{mgw_0} F_{\text{cor}}\right)^2
\end{aligned} \tag{16}$$

where we noted that $mg >> F_{\text{cor}}$ on the Earth, and once again that ω is very small (and so neglecting terms proportional to F_{cor}^2). We can finally find the ratio of rail forces by taking the square root of Equation 16 and plugging in the expression for F_{cor} in terms of g and ω:

$$\boxed{\frac{f_1}{f_2} = 1 + \frac{8\omega z_0}{gw_0} v \sin\theta} \tag{17}$$

(b) Compute value of the ratio.

We're asked to consider a scenario where $v = 100$ m s^{-1}, $\theta = 45$ degrees, $z_0 = 2$ m, $w_) = 1.5$ m, and that $\omega = 7.3 \times 10^{-5}$ rad s^{-1}. We can take $g = 9.806$ m s^{-2}, the local aceeleration due to gravity. With these numbers, we find that

$$\boxed{\frac{f_1}{f_2} \approx 1.006} \tag{18}$$

So, even when deriving Equation 17 by only considering terms that are up to first order in ω, we find that the difference in magnitudes for the two reaction forces are pretty small.

(b) Find the angle of tilt.

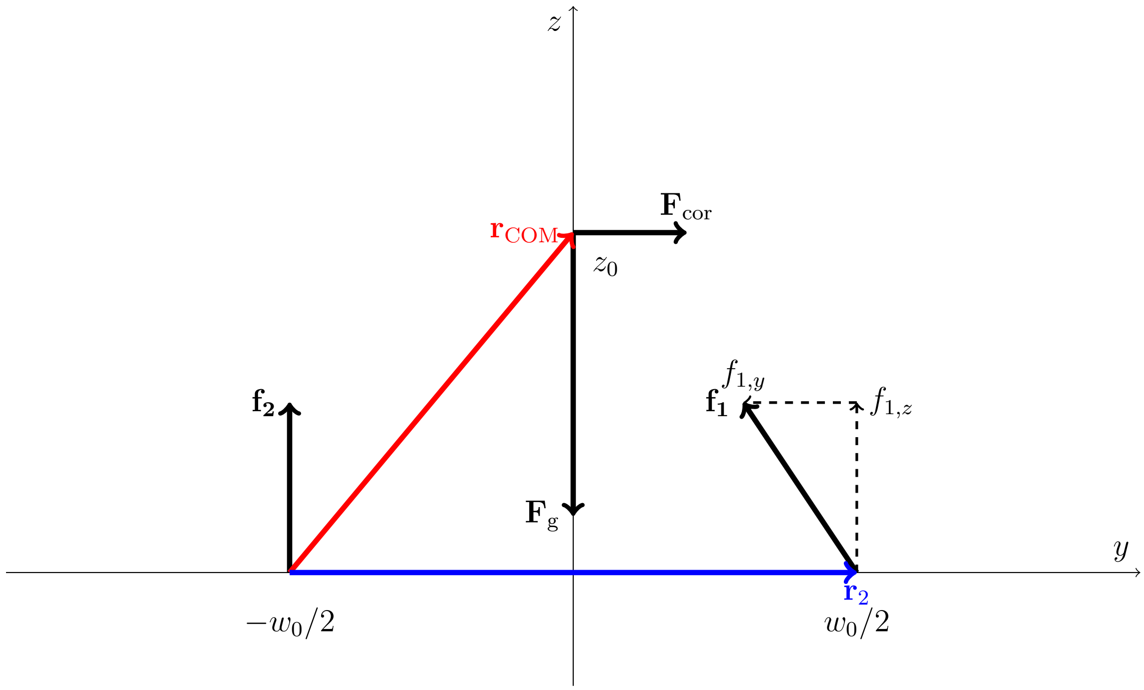
The resultant force due to gravity and the Coriolis effect will only slightly differ from the weight of the train. In other words, we should expect a inclination angle of the platform needed for the resultant force to be perpendicular. We can compute this angle by noting that

$$
\begin{aligned}
\psi &= \arctan\left(\frac{F_{\text{cor}}}{F_g}\right) \\
&= \arctan\left(\frac{2m\omega v \sin\theta}{mg}\right) \\
&= \arctan\left(\frac{2v\omega \sin\theta}{g}\right) \qquad (19)
\end{aligned}
$$

which, when plugging in the numbers from part (b), comes to

$$\boxed{\psi \approx 0.06 \text{ degrees.}} \qquad (20)$$

As expected, this is a very small value, even with the first-order approximation of applicable forces.



z
$\mathbf{F}_{\text{cor}}$
$\mathbf{r}_{\text{COM}}$
z_0
$f_{1,y}$
$\mathbf{f_2}$
$\mathbf{f_1}$
$f_{1,z}$
$\mathbf{F}_{\text{g}}$
y
$\mathbf{r}_2$
$-w_0/2$
$w_0/2$