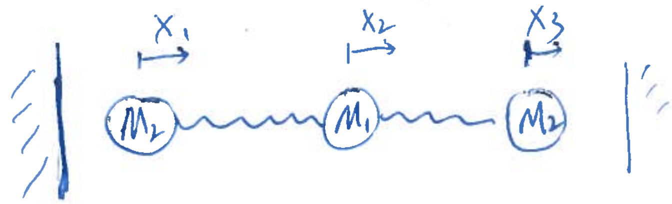
1. (a)



We use the deviation away from equilibrium as our coordinate x_1, x_2, x_3

$$T = \frac{1}{2} m_2 \dot{x}_1^2 + \frac{1}{2} M_1 \dot{x}_2^2 + \frac{1}{2} m_2 \dot{x}_3^2$$

$$V = \frac{1}{2} k (x_1 - x_2)^2 + \frac{1}{2} k (x_2 - x_3)^2$$

$$L = T - V.$$

E.O.M.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} \Rightarrow M_2 \ddot{x}_1 = -k(x_1 - x_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2} \Rightarrow M_1 \ddot{x}_2 = -k(2x_2 - x_1 - x_3)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = \frac{\partial L}{\partial x_3} \Rightarrow M_2 \ddot{x}_3 = -k(x_3 - x_2)$$

The equation for eigenfrequencies is

$$\det\begin{pmatrix} m_2\omega^2-K & +K & 0 \\ K & m_1\omega^2-2K & K \\ 0 & K & m_2\omega^2-K \end{pmatrix}=0$$

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$$m_1 m_2^2\,\omega^6 - 2K(m_1m_2+m_2^2)\,\omega^4 + K^2(2m_2+m_1)\,\omega^2 = 0$$

the solution is

$$\omega^2 = \begin{cases} \dfrac{K}{m_2} \\ \dfrac{m_1+2m_2}{m_1m_2}K, \\ 0 \end{cases}$$

The eigenfrequencies are

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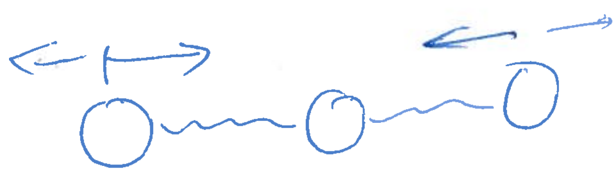
$$\omega_{1,2,3} = \sqrt{\frac{K}{m_2}},\ \sqrt{\frac{m_1+2m_2}{m_1m_2}K},\ 0.$$

b). for $\omega_1 = \sqrt{\frac{k}{m_2}}$, the matrix becomes

$$\begin{pmatrix} 0 & K & 0 \\ K & (\frac{m_1}{m_2}-2)K & K \\ 0 & K & 0 \end{pmatrix}$$

the eigen modes are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

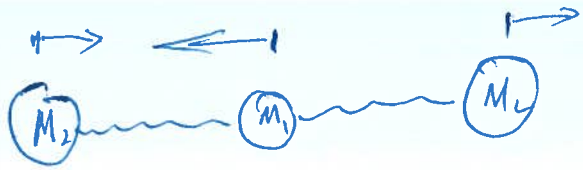


two outer molecules move towards each other.

for $\omega_2 = \sqrt{\frac{m_1+2m_2}{m_1 m_2} K}$, the matrix become

$$\begin{pmatrix} \frac{2m_2}{m_1}K & K & 0 \\ K & \frac{m_1}{m_2}K & K \\ 0 & K & \frac{2m_2}{m_1}K \end{pmatrix}$$

the eigen modes are $\begin{pmatrix} -\frac{m_1}{2m_2} \\ 1 \\ -\frac{m_1}{2m_2} \end{pmatrix}$



The movement is as illustrated.

for $\omega_3 = 0$

$$\begin{pmatrix} -k & +k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix}$$

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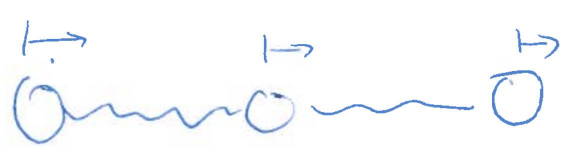
the eigenmodes are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The movement is



All three ~~molecule~~ masses move together

c) The friction is added to the spring. the new E.O.M is

$$\ddot{x}_1 = -\frac{k}{m_2}(x_1 - x_2) - \frac{\eta}{m_2}(\dot{x}_1 - \dot{x}_2)$$

$$\ddot{x}_2 = -\frac{k}{m_1}(2x_2 - x_1 - x_3) - \frac{\eta}{m_1}(2\dot{x}_2 - \dot{x}_1 - \dot{x}_3)$$

$$\ddot{x}_3 = -\frac{k}{m_2}(x_3 - x_2) - \frac{\eta}{m_2}(\dot{x}_3 - \dot{x}_2)$$

assuming our ~~dd.~~eigen modes are X'

we have

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$$\ddot{X} = AX$$

and $TX' = X$ diagonalize A.

a.k.a.

$$\ddot{X}' = T^{-1}ATX' = \begin{pmatrix} -\omega_1^2 & & \\ & -\omega_2^2 & \\ & & -\omega_3^2 \end{pmatrix} X'$$

for the new E.O.M. since the friction is associate with spring. we have

$$\ddot{X} = AX + \frac{\eta}{k} A\dot{X}$$

so $TX' = X$ still diagonalize the equation.

$$\ddot{X}' = \begin{pmatrix} -w_1^2 & & \\ & -w_2^2 & \\ & & -w_3^2 \end{pmatrix} X' + \frac{\eta}{k} \begin{pmatrix} -w_1^2 & & \\ & -w_2^2 & \\ & & -w_3^2 \end{pmatrix} \dot{X}'$$

so all 3 modes still remain decoupled.

2. The equation of motion with external force is

$$\ddot{x} = -\omega_0^2 x - \eta \dot{x} + \frac{F(t)}{m}$$

** ($\eta \to \gamma$ for all bellow)

We define the Fourier transform of the external force is

$$F(\Omega) = \int_{-\infty}^{+\infty} F(t) e^{-i\Omega t} dt$$

and $$F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\Omega) e^{i\Omega t} d\Omega .$$

Fourier Transform the E.O.M. we have

$$X(\Omega) = \frac{1}{\omega_0^2 - \Omega^2 + i\eta\Omega} \cdot \frac{F(\Omega)}{m}$$

Inverse transform $X(\Omega)$ gives a particular solution ~~to~~ E.O.M.

~~Then~~ To do the inverse Fourier transform

$$X(t) = \frac{1}{2\pi M}\int_{-\infty}^{+\infty} \frac{1}{w_0^2 - \Omega^2 + i\eta\Omega} F(\Omega)\, e^{i\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi M}\int_{-\infty}^{+\infty} d\Omega \int_{-\infty}^{+\infty} dt' \frac{1}{w_0^2 - \Omega^2 + i\eta\Omega} F(t')\, e^{-i\Omega t'} e^{i\Omega t}$$

$$= \frac{1}{2\pi m}\int_{-\infty}^{+\infty} d\Omega \int_{-\infty}^{+\infty} dt' \frac{F(t')\, e^{i\Omega(t-t')}}{w_0^2 - \Omega^2 + i\eta\Omega}$$

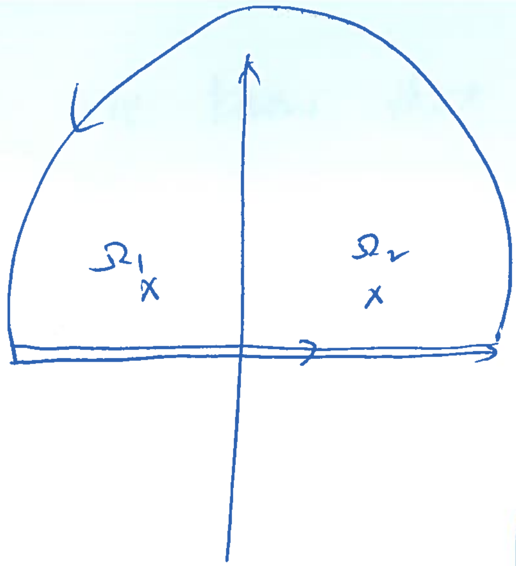
We do the integral over $d\Omega$ first

$$\int_{-\infty}^{+\infty} d\Omega \frac{e^{i\Omega(t-t')}}{w_0^2 - \Omega^2 + i\eta\Omega}$$

$$= \int_{-\infty}^{+\infty} d\Omega \frac{-e^{i\Omega(t-t')}}{(\Omega - \Omega_1)(\Omega - \Omega_2)}$$

$$\Omega_{1,2} = \frac{i\eta}{2} \pm \sqrt{w_0^2 - \frac{\eta^2}{4}}$$

if $4w_0^2 \geq \eta^2$, then both $\Omega_{1,2}$ are in the upper half ~~plane~~ complex plane.



So the integral is only nonzero when $t > t'$.

if $t > t'$, we can use residue theorem to get

$$\int_{-\infty}^{+\infty} d\Omega \, \frac{-e^{i\Omega(t-t')}}{(\Omega-\Omega_1)(\Omega-\Omega_2)}$$

$$= \frac{-2\pi i}{\Omega_1-\Omega_2}\left(e^{i\Omega_1(t-t')} - e^{-i\Omega_2(t-t')}\right)$$

$$= \frac{2\pi}{\sqrt{w_0^2-\frac{\eta^2}{4}}}\, e^{-\frac{\eta}{2}(t-t')} \sin\left(\sqrt{w_0^2-\frac{\eta^2}{4}}\,(t-t')\right)$$

So. $$x(t) = \frac{1}{2\pi m}\int_{-\infty}^{+\infty} dt' \, \frac{\Theta(t-t')\cdot 2\pi}{\sqrt{w_0^2-\frac{\eta^2}{4}}}\, e^{-\frac{\eta}{2}(t-t')} \sin\left(\sqrt{w_0^2-\frac{\eta^2}{4}}\,(t-t')\right) F(t')$$

$$= \frac{1}{m\sqrt{w_0^2-\frac{\eta^2}{4}}}\int_{-\infty}^{t} dt' \, e^{-\frac{\eta}{2}(t-t')} \sin\left(\sqrt{w_0^2-\frac{\eta^2}{4}}\,(t-t')\right) F(t')$$

We know that $F(t) = \Theta(t) F_0 e^{-At} \cos\omega t$.

So $$x_s(t) = \frac{1}{m\sqrt{\omega_0^2 - \frac{\eta^2}{4}}} \int_0^t dt' \, e^{-\frac{\eta}{2}(t-t')} \sin\left(\sqrt{\omega_0^2-\frac{\eta^2}{4}}\,(t-t')\right) \cdot e^{-At'} \cos\omega t'$$

$$= \frac{e^{-\frac{\eta}{2}t}}{m \cdot 2\sqrt{\omega_0^2-\frac{\eta^2}{4}}}\left[\frac{\left(\frac{\eta}{2}-A\right)\left[e^{\frac{\eta}{2}-A}\sin\omega t - \sin\left(\sqrt{\omega_0^2-\frac{\eta^2}{4}}\,t\right)\right]}{\left(\frac{\eta}{2}-A\right)^2 + \left(\omega - \sqrt{\omega_0^2-\frac{\eta^2}{4}}\right)^2} \right.$$

$$+ \frac{\left(\omega - \sqrt{\omega_0^2-\frac{\eta^2}{4}}\right)\cos\left(\sqrt{\omega_0^2-\frac{\eta^2}{4}}\,t\right)}{\left(\frac{\eta}{2}-A\right)^2 + \left(\omega - \sqrt{\omega_0^2-\frac{\eta^2}{4}}\right)^2} -$$

$$\left. \frac{\left(\frac{\eta}{2}-A\right)\left[e^{\frac{\eta}{2}-A}\sin\omega t + \sin\left(\sqrt{\omega_0^2-\frac{\eta^2}{4}}\,t\right)\right] + \left(\omega + \sqrt{\omega_0^2-\frac{\eta^2}{4}}\right)\cos\left(\sqrt{\omega_0^2-\frac{\eta^2}{4}}\,t\right)}{\left(\frac{\eta}{2}-A\right)^2 + \left(\omega + \sqrt{\omega_0^2-\frac{\eta^2}{4}}\right)^2} \right]$$

which is the special solution

the ~~general~~ homogeneous solution

is $x_g(t) = C_0 e^{-\frac{\eta}{2}t} \cos\left(\sqrt{\omega_0^2 - \frac{\eta}{4}}\, t + \varphi\right)$

C, φ are constant

The sollution is

$$x_{\text{[illegible]}}(t) = x_s(t) + x_g(t)$$