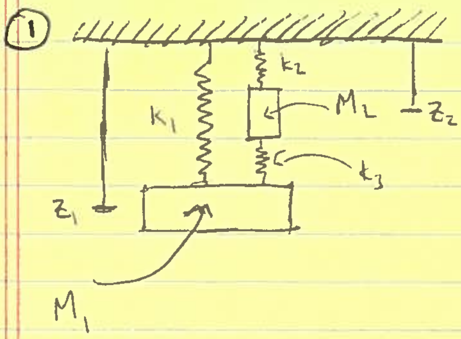
ASSIGNMENT #2 : MODEL ANSWER

(1) 

A system of vertically-coupled oscillators is shown on the left. The interaction between the masses M_1 and M_2 is mitigated by a third spring that connects them, with spring constant k_3

a. Find the Lagrangian for this system: let the displacements of masses M_1 and M_2 be z_1 and z_2, respectively. Then the total kinetic energy is

$$T = T_1 + T_2 = \frac{1}{2} M_1 \dot{z}_1^2 + \frac{1}{2} M_2 \dot{z}_2^2$$

and the total potential, including the coupling term, is

$$V = \frac{1}{2} k_1 z_1^2 + \frac{1}{2} k_2 z_2^2 + \frac{1}{2} k_3 (z_1 - z_2)^2 - m_1 g z_1 - m_2 g z_2$$

so that the full Lagrangian is $L = T - V$. So, now apply Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_1}\right) - \frac{\partial L}{\partial z_1} = M_1 \ddot{z}_1 + k_1 z_1 + k_3 (z_1 - z_2) - M_1 g = 0$$

$$= M_1 \ddot{z}_1 + (k_1 + k_3) z_1 - k_3 z_2 - M_1 g \longrightarrow \boxed{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_2}\right) - \frac{\partial L}{\partial z_2} = M_2 \ddot{z}_2 + (k_2 + k_3) z_2 - k_3 z_1 - M_2 g = 0 \longrightarrow \boxed{2}$$

So equations $\boxed{1}$ and $\boxed{2}$ have the equivalent forms

$$\ddot{z}_1 + (\omega_1^2 + k_3/M_1) z_1 - (k_3/M_1) z_2 = M_1 g \longrightarrow \boxed{3}$$

$$\ddot{z}_2 + (\omega_2^2 + k_3/M_2) z_2 - (k_3/M_2) z_1 = M_2 g \longrightarrow \boxed{4}$$

and these two equations are similar in form to Equation (10) in the "Coupled Oscillators" notes on the course website. Here, $\omega_1^2 = k_1/M_1$ and $\omega_2^2 = k_2/M_2$. We will therefore use the same method to define new variables and re-write the above equations ($\boxed{3}$ and $\boxed{4}$) so that they can be solved.

Now, let's define $\lambda_1 = k_3/M_1$, $\lambda_2 = k_3/M_2$, $\Omega_1^2 = \omega_1^2 + \lambda_1$, $\Omega_2 = \omega_2^2 + \lambda_2$, $f_1 = M_1 g$, $f_2 = M_2 g$, so that we get the equations

$$\ddot{z}_1 + \Omega_1^2 z_1 - \lambda_1 z_2 = f_1$$

$$\ddot{z}_2 + \Omega_2^2 z_2 - \lambda_2 z_1 = f_2$$

now use Equations (12) and (13) from the "Coupled Oscillator" course notes to define new time-dependent variables:

Let $x_1 = z_1 - \bar{z}_1$ and $x_2 = z_2 - \bar{z}_2$, where

$$\bar{z}_1 = \frac{g_2 a_1 + a_2}{g_1 g_2 - 1}, \quad \bar{z}_2 = \frac{g_1 a_2 + a_1}{g_1 g_2 - 1}$$

and the constants here are

$$g_1 = \frac{k_3}{M_1 \Omega_1^2}, \quad g_2 = \frac{k_3}{M_2 \Omega_2^2}, \quad a_1 = -f_1, \quad \text{and } a_2 = -f_2$$

and now redefine x_1 and x_2 $\longrightarrow$ let $x_1 = y_1$ and $x_2 = (\lambda_2/\lambda_1)^{1/2} y_2$, so that we finally get

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$$\boxed{\begin{aligned} \ddot{y}_1 + \Omega_1^2 y_1 - \lambda y_2 &= 0 \\ \ddot{y}_2 + \Omega_2^2 y_2 - \lambda y_1 &= 0 \end{aligned}}$$

where $\lambda = (\lambda_1 \lambda_2)^{1/2}$

these equations of motion can be obtained from a Lagrangian of the following form:

$$L(y_1, y_2, \dot{y}_1, \dot{y}_2) = \frac{1}{2}(\dot{y}_1^2 + \dot{y}_2^2) + \frac{1}{2}(\Omega_1^2 y_1^2 + \Omega_2^2 y_2^2) - \lambda y_1 y_2$$

which is the same form of a Lagrangian that describes two simple harmonic oscillators with (uncoupled) frequencies Ω_1 and Ω_2, but with an additional bilinear term in the potential that couples the two coordinates together.

b. Solve for the two eigenfrequencies of the system

In order to find the eigenfrequencies, we must assume that the solutions of the equations of motion have the following form

$$y_1 = A_1 \exp(i\omega t), \quad y_2 = A_2 \exp(i\omega t) \qquad \boxed{6}$$

where $A_1 \neq A_2$ are complex amplitudes. So we plug in Equation (6) to Equation (5) to get:

$$-\omega^2 A_1 \exp(i\omega t) + \Omega_1^2 A_1 \exp(i\omega t) - \lambda A_2 \exp(i\omega t) = 0 \qquad \boxed{7}$$

$$-\omega^2 A_2 \exp(i\omega t) + \Omega_2^2 A_2 \exp(i\omega t) - \lambda A_1 \exp(i\omega t) = 0 \qquad \boxed{8}$$

So let's solve Equation (7) to put A_1 in terms of A_2:

$$(\Omega_1^2 - \omega^2) A_1 = \lambda A_2 \implies A_1 = \lambda A_2 (\Omega_1^2 - \omega^2)^{-1} \longrightarrow \boxed{9}$$

Now plug Equation (9) into Equation (8):

$$(\Omega_2^2 - \omega^2)\cancel{A_2} - \lambda^2 \cancel{A_2} (\Omega_1^2 - \omega^2)^{-1} = 0 \implies (\Omega_1^2 - \omega^2)(\Omega_2^2 - \omega^2) - \lambda^2 = 0$$

or, expanding the product term:

$$\omega^4 - (\Omega_1^2 + \Omega_2^2)\omega^2 + \Omega_1^2 \Omega_2^2 - \lambda^2 = 0$$

This is a quadratic equation in ω^2, using the quadratic formula, we get

$$\omega^2 = \frac{(\Omega_1^2 + \Omega_2^2) \pm \sqrt{(\Omega_1^2 + \Omega_2^2)^2 - 4\Omega_1^2\Omega_2^2 + \lambda^2}}{2}$$

$$= \frac{(\Omega_1^2 + \Omega_2^2) \pm \sqrt{\Omega_1^4 + \Omega_2^4 + 2\Omega_1^2\Omega_2^2 - 4\Omega_1^2\Omega_2^2 + \lambda^2}}{2}$$

$$= \frac{(\Omega_1^2 + \Omega_2^2) \pm \sqrt{\Omega_1^2 + \Omega_2^2 - 2\Omega_1^2\Omega_2^2 + \lambda^2}}{2}$$

$$\boxed{\omega^2 = \frac{(\Omega_1^2 + \Omega_2^2) \pm \sqrt{(\Omega_1^2 - \Omega_2^2)^2 + \lambda^2}}{2}}$$

The eigenfrequencies in this problem are ω_+ and ω_-, where

$$\omega_+^2 = \frac{(\Omega_1^2 + \Omega_2^2) + \sqrt{(\Omega_1^2 - \Omega_2^2)^2 + \lambda^2}}{2}$$

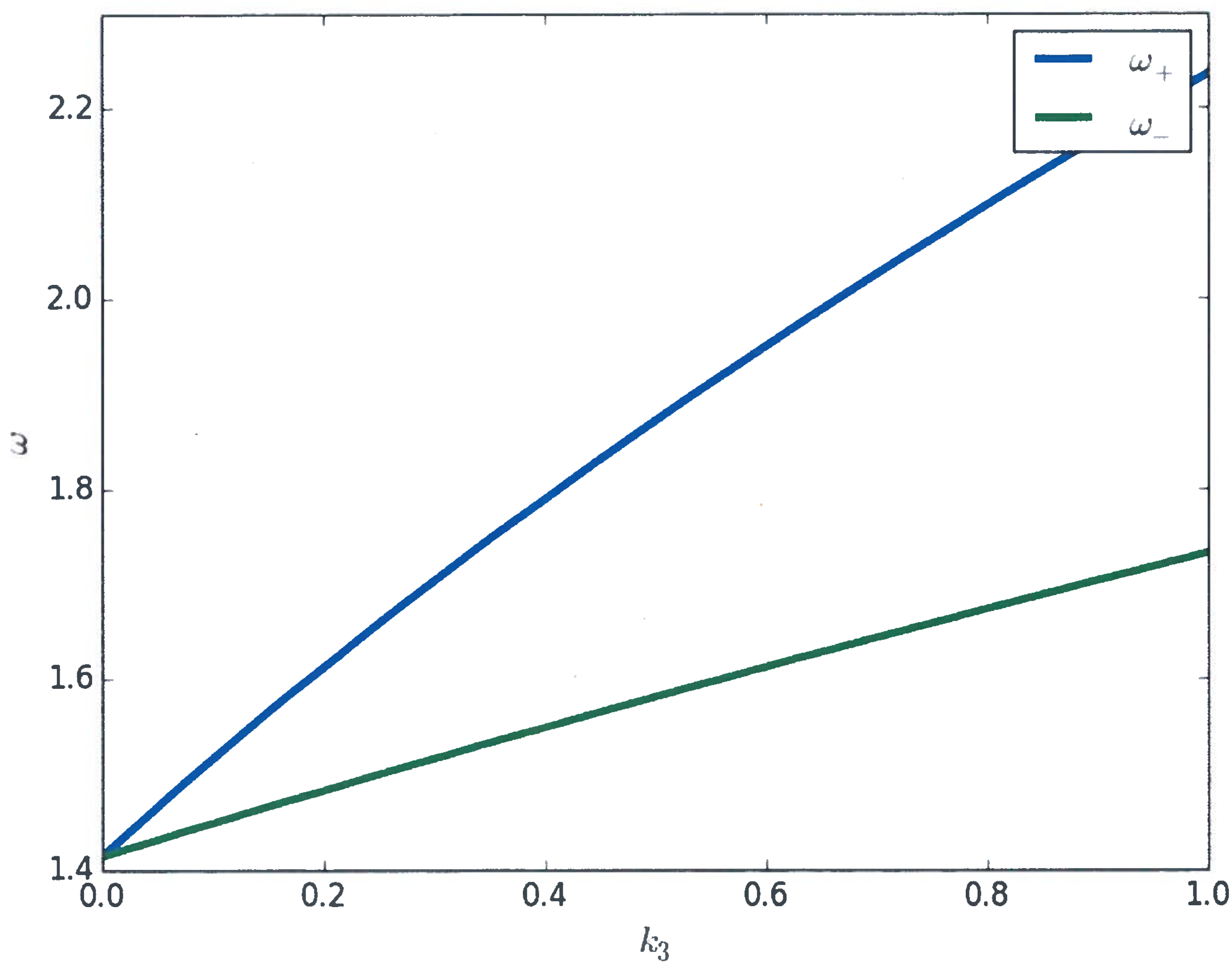
$$\omega_-^2 = (\Omega_1^2 + \Omega_2^2) - \sqrt{(\Omega_1^2 - \Omega_2^2) + \lambda^2}$$

and in both cases, $\omega_+ > 0$ and $\omega_- > 0$, so we take the positive roots.

Problem 1 - eigenfrequency vs. k_3

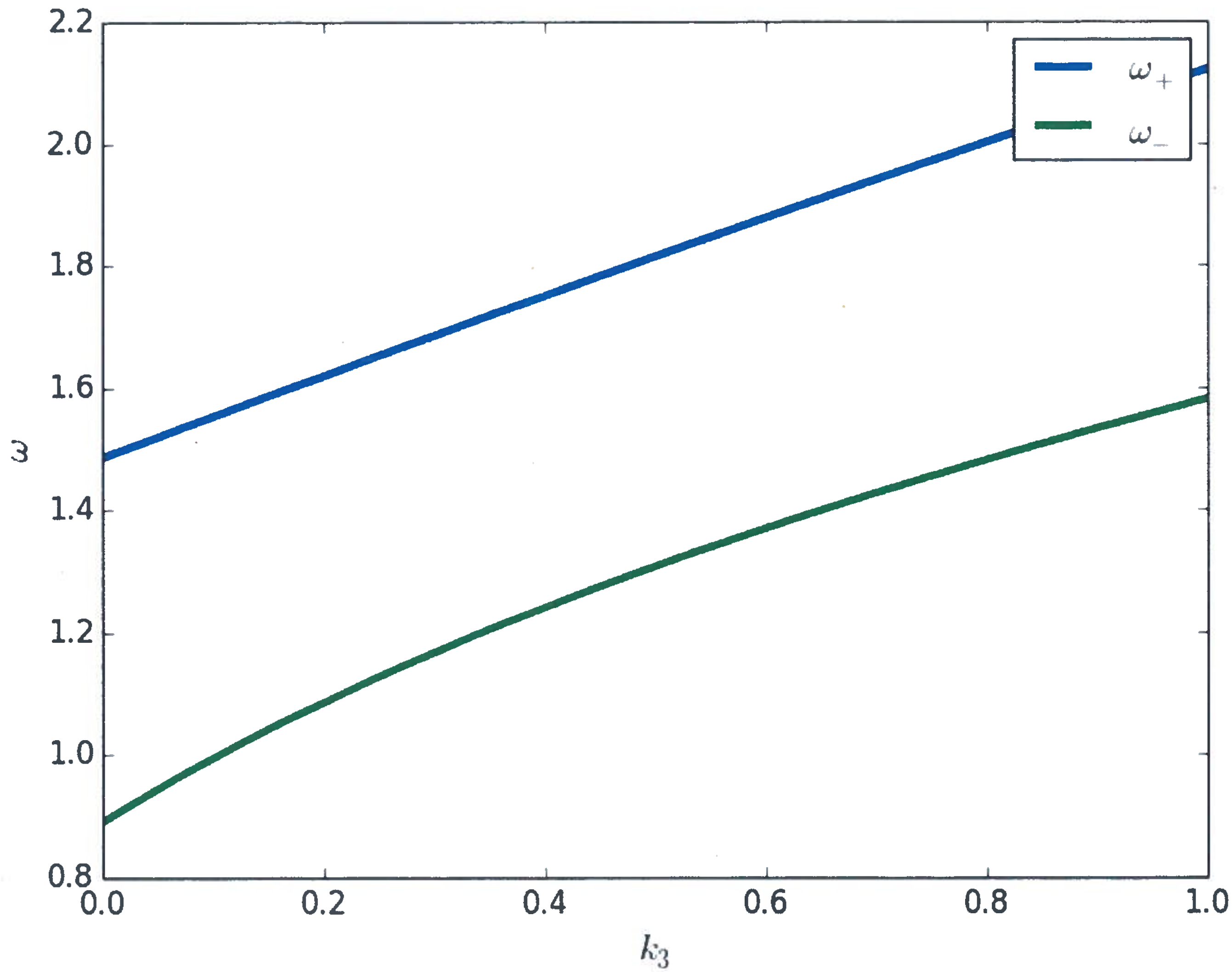
$k_1 = k_2 = k_3 = 1$

and $M_1 = M_2 = 1$



Problem 1 - eigenfrequency vs. k_3

$k_1 = k_2 = k_3 = 1$, $M_1 = 1$, $M_2 = 2$



② Consider a two-dimensional oscillator, with coordinates q_1 and q_2, and a Lagrangian

$$L = \frac{1}{2}\left[(\dot{q}_1^2 + \dot{q}_2^2) - (\omega_1^2 q_1^2 + \omega_2^2 q_2^2) - g q_1 q_2\right]$$

<u>a. Find equations of motion and solve for the eigenfrequencies of this system:</u>

Euler-Lagrange equations: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$, for $j = 1, 2$

so for q_1: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \boxed{\ddot{q}_1 + \omega_1^2 q_1 + \frac{1}{2} g q_2 = 0}$

so for q_2: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \boxed{\ddot{q}_2 + \omega_2^2 q_2 + \frac{1}{2} g q_1 = 0}$

These are coupled, symmetric differential equations; to solve them, assume a form of the oscillatory motion: $q_1 = A_1 \exp(i\omega t)$ and $A_2 \exp(i\omega t) = q_2$, and so the differential equations become

$$\begin{cases} -\omega^2 A_1 \cancel{\exp(i\omega t)} + \omega_1^2 A_1 \cancel{\exp(i\omega t)} + \frac{1}{2} g A_2 \cancel{\exp(i\omega t)} = 0 \\ -\omega^2 A_2 \cancel{\exp(i\omega t)} + \omega_2^2 A_2 \cancel{\exp(i\omega t)} + \frac{1}{2} g A_1 \cancel{\exp(i\omega t)} = 0 \end{cases}$$

$\Longrightarrow A_1(\omega_1^2 - \omega^2) = -\frac{1}{2} g A_2 \Longrightarrow A_1 = g A_2 / (\omega_1^2 - \omega^2)$, substitute

$\Longrightarrow \cancel{A_2}(\omega_2^2 - \omega^2) = -\frac{1}{2} g A_1 = g^2 \cancel{A_2} (\omega_1^2 - \omega^2)^{-1}$, and so

$(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = \frac{1}{4} g^2$, so let $x = \omega^2$ and we get

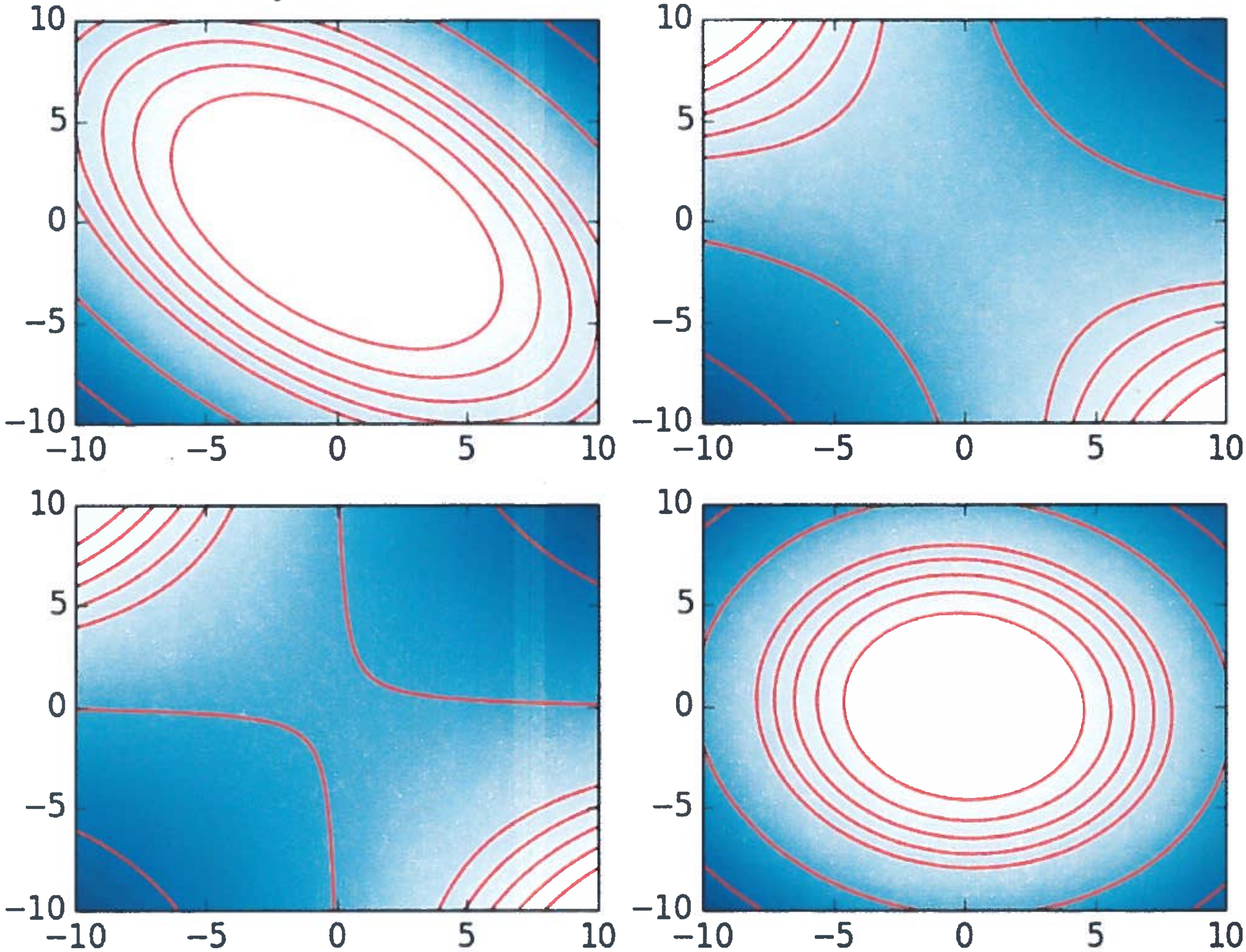
$x^2 - (\omega_1^2 + \omega_2^2)x + \omega_1^2 \omega_2^2 - \frac{1}{4} g^2 = 0$, so the solution is

$$x = \boxed{\omega^2 = \frac{(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4(\omega_1^2 \omega_2^2 - \frac{1}{4} g^2)}}{2}}$$

Problem 2 - contours for different values of g

In all cases, $w_1 = w_2 = 1$

$g = 1.0$ $g = 10$



$g = 100$ $g = 0.1$

Problem 2 - eigenfrequency vs. g

$\omega_1 = \omega_2 = 1$