

# PHYS 306: MOCK EXAM

(composed March 28, 2016)

To be done whenever you like

This is a MOCK exam. The real exam will last 2 hrs and 30 mins. The only material allowed into the real exam will be pens, pencils, and erasers. No notes of any kind will be permitted, nor any calculators or other electronic devices.

There are 2 sections. In the real exam, students should answer THREE QUESTIONS ONLY from section A, and TWO QUESTIONS ONLY from section B. No extra marks will be given in the real exam for extra questions answered. The questions in section A should take roughly 15-20 minutes to answer, and the questions in section B roughly 45-50 minutes to answer.

In doing this mock exam, you should pretend it is a real exam - ie., no cheat sheets, or books, or anything else to help you.

---

## SECTION A

**A1:** Consider a 1-dimensional potential given as a function of position  $r$  by

$$U(r) = \frac{V_P}{r^{12}} - \frac{V_W}{r^6}$$

Find the value  $r_o$  of  $r$  for which  $U(r)$  is a minimum, and expand about  $r_o$  to get the harmonic potential in its vicinity. Hence determine the frequency of oscillation about this minimum, of a mass  $m$ .

**A2:** The gravitational potential of the earth acting on a mass  $m$  is  $V(r) = -GM_o m/r$ , where  $M_o$  is the mass of the earth. If the radius of the earth is  $R_o$ , find the gravitational force  $mg$  acting on a mass at the earth's surface in terms of  $R_o$ . The escape velocity  $v_o$  is defined as that velocity for which the total energy (kinetic plus potential) is zero. Find  $v_o$  as a function of  $R_o$  and  $g$ . Now, assuming that  $R_o \sim 6 \times 10^3$  km, and  $g \sim 10 \text{ ms}^{-2}$ , find an approximate answer for  $v_o$  for the earth (you will *not* need a calculator for this!!).

**A3:** Consider a cylindrical solid of uniform density  $\rho_o$ , radius  $R_o$ , and length  $L$ . Calculate the rotational energy of the object if it is rotating about the cylindrical axis with angular velocity  $\omega$ . Now, suppose the cylinder rolls without slipping down an inclined plane. What fraction of the total kinetic energy of the cylinder is in its rotational energy?

**A4:** A particle of mass  $M$  is constrained to move on a vertically oriented circle of radius  $R_o$  - it is also attached by a spring with spring constant  $k$  to a point at a distance  $2R_o$  above the centre of the circle.

Write down the Lagrangian for the mass, and find the equation of motion in the case where it only moves a small angle away from a position vertically above the centre of the circle.

**A5:** The moment of inertia tensor is typically written in component form, in Cartesian coordinates, for a continuous solid with mass density distribution  $\rho(\mathbf{r})$ , as

$$I_{\alpha\beta} = \int d^3r \rho(\mathbf{r})(r^2 \delta_{\alpha\beta} - r^\alpha r^\beta)$$

Write out the components of  $I_{\alpha\beta}$  in the form of a  $3 \times 3$  matrix.

**A6:** A mass  $M$  is attached to a point by a spring of spring constant  $k$ , and is otherwise free to move without friction in a horizontal plane.

Write down the Lagrangian for the motion of the mass in the plane, and derive the equation of motion for the radial coordinate  $r$  (the distance from the centre).

---

## SECTION B

**B1: CENTRAL FIELD MOTION:** We consider a mass  $m$  moving in a central force potential  $V(r)$ , where  $r$  is the radial coordinate of the vector  $\mathbf{r} = (r, \phi)$ , and we assume motion in the  $(r, \phi)$  plane.

(i) Write down an expression for the total energy  $E$  of the particle, in terms of  $\dot{r}$ ,  $\dot{\phi}$ , and  $r$ . Then rewrite this in terms of  $\dot{r}$  and  $r$  only, by using the conservation of angular momentum  $L$  about the central point at  $r = 0$ . Since  $E$  is also conserved, you should now also be able to write an expression - an equation of motion - for  $\dot{r}$  in terms of  $E, L, r, m$  and the potential  $V(r)$ .

(ii) To solve the equation of motion, it is convenient to change variables to  $y = 1/r$ . Rewrite  $\dot{r}$  in terms of  $dy/d\phi$  and the angular momentum  $L$ , using  $\dot{r} = \dot{\phi}(dr/d\phi)$ . Then substitute this into the equation of motion for  $\dot{r}$  you derived above, and show that the

variable  $y(\phi)$  obeys the following equation of motion:

$$y^2 + \left(\frac{dy}{d\phi}\right)^2 = \frac{2m}{L^2}(E - V(r))$$

no matter what the potential  $V(r)$  may be.

**(iii)** Let us now assume that the potential  $V(r) = V_0 r^2/2$ . Draw a graph, as a function of  $r$ , of the effective radial potential that results from summing  $V(r)$  and the centrifugal potential. Then, by minimizing this potential as a function of  $r$ , for some given angular momentum  $L$ , find the radius  $r_o$  of the orbit for which the system exhibits circular motion. What is the period of this motion?

**(iv)** Finally, by examining the form of the effective radial potential around  $r_o$ , derive the period of the small radial oscillations about a circular orbit, when the energy  $E$  is very slightly above the minimum energy for a given  $L$ .

**B2: COUPLED OSCILLATORS:** We consider a pair of LC circuits, coupled by a mutual capacitance. The Lagrangian for this can be written down by noting that the energy in a current flowing through an inductance  $L$  is  $LI^2/2$ , where  $I = dQ/dt$  is the current; the energy stored in a capacitance  $C$  is  $Q^2/2C$ ; and the energy in the mutual capacitance  $C_{12}$  between two LC circuits is  $Q_1 Q_2 / C_{12}$ , where  $Q_1, Q_2$  are the charges on the 2 capacitors in each of the 2 circuits.

**(i)** Write down an expression for the Lagrangian of the total system, in terms of the  $Q_j(t)$  and  $\dot{Q}_j(t)$ , where  $j = 1, 2$ .

**(ii)** Find the equations of motion of this coupled LC circuit, and solve them assuming that the initial currents in the system are zero, and that the initial charges are  $\bar{Q}_1, \bar{Q}_2$ .

**(iii)** Now suppose we insert resistors with resistances  $R_1$  and  $R_2$  respectively into the 2 LC circuits. Using Ohm's law, that the voltage "force" across a resistor is  $V = IR$ , find the new equations of motion for the coupled circuits.

**(iv)** Now suppose that we let the resistance  $R_2 \rightarrow \infty$ . Show from the equations of motion that the current in circuit 2 is then zero, so that the charge  $Q_2$  is frozen to be  $\bar{Q}_2$  for all time. Then from this write down the equation of motion for the charge  $Q_1(t)$  in the first circuit, and solve it using the same initial conditions as you had in part (ii) above.

**B3. ROTATING SPACE STATION:** A large space station is approximately described as having 2 parts: a long massive cylindrical spindle of length  $L_o$ , mass  $M_o$ , and negligible radius; and a large circular ring concentric with the centre of mass of the spindle (and joined to it by "spokes"), of radius  $R_o$ , and mass  $M_o$ ; we assume that  $m \ll M_o$ , and that the thickness of the ring is also negligible.

(i) Find the 3 principal moments of inertia, about the centre of mass of the system, as functions of  $M_o, m, L_o$ , and  $R_o$ .

(ii) Derive the centrifugal force acting on some object of very small mass  $m$  situated on the ring, if the space station is rotating about the cylindrical axis at angular velocity  $\omega$ . Then, assuming that the radius  $R_o = 100 m$ , find out what  $\omega$  must be for the mass to feel a centrifugal force  $mg$ , where  $g = 10ms^{-2}$  is the acceleration experienced at the earth's surface.

(iii) Suppose now that we launch a cylindrical spaceship of mass  $m_s$ , and radius  $r_o$  from the centre of the central spindle, along the spindle axis with acceleration  $a$ ; we assume the spaceship's axis is also aligned along the spindle axis, and that it is already spinning about this axis with angular velocity  $\omega$ .

Describe quantitatively the path that a spot of paint on the outer cylindrical surface of the spacecraft would appear to follow as seen by an outside inertial observer; and show a diagram of this path. At what velocity would the spacecraft be traveling as it left the end of the spindle?

(iv) Inside the spacecraft, a ball is thrown outwards by one of the astronauts radially towards the outer surface of the spacecraft, at a velocity  $v$ . Find the path that this ball would appear to follow (a) for an observer inside the spacecraft, and (b) for an inertial observer outside the craft.

**B4. TIDAL FORCES** Consider a long thin rod, of mass density  $\pi r_o^2 \rho$  per unit length, and total length  $l_o$ ; we assume that  $r_o$  is negligible. It is situated at a distance  $R_o$  from the centre of the earth, and the rod axis is oriented at an angle  $\theta$  with respect to the radial vector from the centre of the earth out to the rod.

(i) The different parts of the rod are attracted differently to the earth, depending on their distance from the earth - the energy of the rod then depends on its orientation angle  $\theta$ .

Assuming the gravitational potential energy of a unit point mass at a distance  $r$  from the earth is  $-GM_o/r$ , where  $M_o$  is the mass of the earth, calculate the total potential energy of the rod as a function of the angle  $\theta$ , by integrating along the rod, and show that is given

approximately, when  $l_o \ll R_o$ , by

$$V(R_o, \theta) \sim -G \frac{M_o m_o}{R_o} \left[ 1 + \frac{l_o^2}{6R_o^2} \cos^2 \theta \right]$$

where  $m_o$  is the mass of the rod (Hint: expand in powers of  $l_o/R_o$ ).

(ii) If the rod is rotating, it will have a rotational kinetic energy proportional to  $\dot{\theta}^2$ . Calculate this rotational kinetic energy, and hence write down the Lagrangian for the rotational motion of the rod.

(iii) Find the frequency of small oscillations of the rod about the vertical position  $\theta = 0$  (or  $\theta = \pi$ ).

(iv) Assume now that the rod is moving in a circular orbit about the earth. Using any method you like, derive the period of revolution of the rod around the earth. For what value of  $R_o$  will this period match the period of angular oscillations of the rod?

**END of MOCK EXAM**