PHYS 306: FINAL EXAM

(Friday, April 22nd, 2016)

12.00 - 14.30 pm, Room IBLC 261

This exam will last 2 hrs and 30 mins. The only material allowed into the exam will be pens, pencils, and erasers. No notes of any kind are permitted, nor any calculators or other electronic devices.

There are 2 sections. Students should answer THREE QUESTIONS ONLY from section A, and TWO QUESTIONS ONLY from section B. No extra marks will be given for extra questions answered. The questions in section A should take roughly 15-20 minutes to answer, and the questions in section B roughly 45-50 minutes to answer.

SECTION A

A1: Show that a vector $\mathbf{A}(t)$ has a total time derivative $d\mathbf{A}/dt$ in a rotating non-inertial frame given by $d\mathbf{A}/dt = \partial\mathbf{A}/\partial t + (\omega \times \mathbf{A})$, where $\partial\mathbf{A}/\partial t$ is the time derivative in the non-rotating inertial frame, and ω is the angular velocity of rotation of the non-inertial frame.

Now, using this result, show that the second time derivative of $\mathbf{A}(t)$ in the rotating non-inertial frame is given by

$$\frac{d^2\mathbf{A}}{dt^2} \quad = \quad \frac{\partial^2\mathbf{A}}{\partial t^2} \; + \; 2\left(\omega \times \frac{\partial\mathbf{A}}{\partial t}\right) \; + \; \left(\frac{\partial\omega}{\partial t} \times \mathbf{A}\right) \; + \; \omega \times (\omega \times \mathbf{A})$$

A2: Two objects of mass m moving on a line are connected by a spring with spring constant k; but they are also electrically charged, creating a repulsive potential $V(x) = V_o/x$ between them.

Write down the total potential between the two masses, and find the point of equilibrium where the net force between them is zero. By expanding the potential about this point, find the frequency of small oscillations for the masses.

A3: Suppose the 2-bladed propeller of a plane can be treated as a thin rod of length 2 m, and a mass of 300 kg. Calculate the moment of inertia of the propeller about its axis of rotation (the centre of the rod), assuming its mass is distributed uniformly along its length, and ignoring its thickness. Now suppose it is rotating at an angular velocity of 1000 rads/sec; what are its rotational kinetic energy and angular momentum? What is the velocity of the ends of the propeller blades? And if all the rotational energy were converted to centre of mass motion, how fast would the centre of mass move? Give the correct MKS units for all quantities you find.

A4: The action of a system is defined as $S = \int_{t_1}^{t_2} d\tau \mathcal{L}(q, \dot{q})$. By assuming that for the correct classical trajectory, S is stationary with respect to variations of $q(\tau)$ and $\dot{q}(\tau)$, derive Lagrange's equations of motion for the system.

A5: Write down the Lagrangian of a system of mass m orbiting in a central field potential $V(r) = -V_o/r$, with $V_o > 0$. Then show, by any means you like, that for circular motion one has $mu^2/r = V_o/r^2$, where u is the orbital velocity.

A6: Two springs, with spring constants k_1 and k_2, are connected to masses m_1 and m_2, having coordinates q_1, q_2. The masses are also coupled to each other by an interaction $\kappa_o q_1 q_2$; and there is a force $F_1(t)$ acting on the first spring. Write down the Lagrangian for the system, and then find the equations of motion (no need to solve them!).

SECTION B

B1: CENTRAL FIELD MOTION: An object of mass m is moving in a central field potential $V(r)$. We suppose the initial velocity of the mass is in the plane perpendicular to the z-axis.

(i) Write down the Lagrangian $\mathcal{L}$, the total energy E, and the angular momentum L for the object, as functions of the radial coordinate r, the azimuthal angle ϕ, and their time derivatives. Then derive the equations of motion for r and ϕ using Lagrange's equations; and using the conservation of L and E, rewrite these so that the equation of motion for r, linear in $\dot{r}$, is independent of that for ϕ.

(ii) We now wish to solve the equation of motion for $r(t)$. Changing variables to $y = 1/r$, you can rewrite $\dot{r}$ in terms of $dy/d\phi$ and the angular momentum L, using $\dot{r} = \dot{\phi}(dr/d\phi)$. Then substitute this into the equation of motion for $\dot{r}$ you derived above, and show that the variable $y(\phi)$ obeys the following equation of motion:

$$y^2 + \left(\frac{dy}{d\phi}\right)^2 = \frac{2m}{L^2}(E - V(r))$$

whatever the potential $V(r)$ may be.

(iii) Let us now assume that the potential $V(r) = -V_o/r^4$, with $V_o > 0$. Draw a graph, as a function of r, of the effective radial potential $U_{eff}(r)$ that results from summing $V(r)$ and the centrifugal potential, assuming that the angular momentum $L > 0$. By looking at radial trajectories that start either at very large r or very small r, you will see that there are 3 possible kinds of radial motion for this system. Explain what these are.

(iv) Now, find the radius r_o for which $U_{eff}(r)$ is a maximum. Suppose that there is a very small amount of friction in the system, which decreases its energy, without appreciably changing L. What do you expect to happen to the particle in this situation?

B2: OSCILLATORS, TIDES, and LUNAR OUTSPIRAL: Tides on earth are mainly caused by the moon's gravitation. We can in principle use them for tidal power; they also cause a slow decay of the earth's rotation, and an 'outspiraling' of the moon from the earth.

(i) To extract the tidal power, we take a large mass M, and attach it to the marine bottom by a spring with spring constant k. At low tide the water depth is z_1, at high tide it is z_2; the mass is kept floating on the water surface using air floats.

What are the MKS units of k and M? Suppose the spring constant is $k = 2.5 \times 10^3$ in these units, and that in the absence of waves, $z_1 = 2\ m$ and $z_2 = 8\ m$. Then (a) what is the downward force on the mass from the spring at high and low tides, and (b) what is the energy stored in the spring at high and low tides?

Suppose now that $M = 10^4$ (in MKS units). What is the natural frequency ω_o of oscillation of the float (give the correct units here)?

Finally, suppose that at high tide, we have time-dependent waves of form $\Delta z(t) = A \cos \omega t$ on the water, giving a float height $z(t) = z_2 + \Delta z(t)$. Assuming that we can extract all the kinetic energy from the float motion, what would be the mean power (averaged over time) that we would get from this? If $A = 2\ m$, how much would this be? Again, give the correct units.

(ii) Almost all of the energy of the earth-moon system is to be found in the angular rotation of the earth, the orbital motion of the moon, and in the earth-moon gravitational potential. Thus we can write the total energy as approximately

$$E \approx \frac{1}{2}mu^2 - \frac{V_o}{r} + \frac{1}{2}I_1\Omega_1^2$$

where m is the lunar mass, u its orbital velocity, r the earth-moon distance, $V_o = GMm$ is the gravitational strength (with M the earth mass), I_1 is the earth's moment of inertia, and

Ω_1 is the earth's rotational angular velocity.

Assume that the moon's orbit is circular, so that $mu^2/r = V_o/r^2$. Then, using this result, show that we can rewrite the energy in the form

$$E = \frac{1}{2}\left(S_1^2 - \frac{mV_o^2}{L^2}\right)$$

where $S_1 = I_1\Omega_1$ is the rotational angular momentum of the earth, and $L = mur$ is the angular momentum contained in the orbital motion of the moon.

(iii) The total angular momentum of the system is $J = L + S_1$ (we neglect the much smaller S_2 coming from the moon's rotational motion), and we can assume that J is conserved, ie., it is constant in time. Write E entirely in terms of L, and then show that the minimum energy E_o, obtained from $dE/dL = 0$, can be written as $L/mr^2 = S_1/I_1$ (here we ignore the other minimum when $r \to 0$). Since $L/mr^2 = \omega$, the lunar orbital angular velocity, and $S_1/I_1 = \Omega_1$, we see the minimum E is attained when the earth's rotational period is locked to the moon's orbital period.

(iv) Currently $\Omega_1 \gg \omega$, so almost all of the kinetic energy of the earth-moon system is in the earth's rotational motion. Draw a graph of $E(L)$ vs. L, and show roughly where you think the earth must now be on this curve.

Kinetic energy is slowly lost through tidal friction. Show, from the definition of L, that $\dot{L}/L = \dot{r}/2r$, and thus, since J is constant, that

$$\dot{r}/r = -2\dot{S}_1/S_1 = -2\frac{\dot{\Omega}}{\Omega}\frac{S_1}{L}$$

Thus as S_1 decreases, L increases, and so does r. This is the outspiraling of the moon; most of the earth's rotational energy goes into increasing the moon's orbital radius.

B3: TABLE TENNIS BAT: We will look here at the dynamics of a table tennis bat, which has 3 different moments of inertia I_1, I_2, I_3 along its principal axes. The bat is made from a thin circular plate of wood, of diameter 16 cm, and mass 300 g, attached at its edge to a solid cylinder (the handle) of wood, of length 14 cm and mass 150 g, oriented so that its extension along its axis would pass through the centre of the circular plate.

(i) Draw the bat, and then calculate the position of the centre of mass. Now, draw diagrams showing the rotations about the 3 principal axes. Without trying to calculate the moments of inertia I_j about these 3 axes, which do you think is the largest, and which do you think is the smallest?

(ii) The angular momentum vector $\mathbf{L}(t)$ depends on the torque $\mathbf{N}(t)$ according to the equation $d\mathbf{L}/dt = \mathbf{N}(t)$, in an inertial frame. From this, derive Euler's equations for the dynamics of the angular velocity $\mathbf{\Omega}(t)$ in the *rotating frame*, in the form

$$\begin{aligned} I_1\dot{\Omega}_1 + (I_3 - I_2)\Omega_3\Omega_2 &= N_1 \\ I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_1\Omega_3 &= N_2 \\ I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_2\Omega_1 &= N_3 \end{aligned} \tag{1}$$

where the Ω_j and the N_j are the 3 components of the vectors $\mathbf{\Omega}$ and $\mathbf{N}$ along the 3 principal axes $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$.

(iii) Let's now assume that the ordering in size of the moments of inertia is $I_1 > I_2 > I_3$; and assume also that the angular velocity is directed very nearly along the $\hat{e}_2$-axis; hence $\Omega_2 \gg \Omega_1, \Omega_3$, so that we can neglect the product $\Omega_1\Omega_3$ in the Euler equations.

Suppose the bat has been thrown in the air, and is now in free fall. Show that in this case $\Omega_2 \sim$ constant. Then solve the 2 coupled equations for $\Omega_1(t)$ and $\Omega_3(t)$ (you will need to write your answer in terms of arbitrary constants referring to the initial conditions). Show that your result implies that the motion about the axis $\hat{e}_2$ is unstable.

B4. ROLLING CYLINDER PENDULUM

We are going to compare 2 different "pendulums". The first involves a mass M moving without friction along the perimeter of a circle of radius l_o,; the circle is oriented vertically (ie., with the axis through its centre oriented horizontally), so that the mass M moves in a vertical plane. The second pendulum has a solid cylinder, of uniform density ρ_o and radius r_o, also with mass M, which is rolling without slipping and without friction along the same circular path. We assume $r_o \ll l_o$, so that the centre of the cylinder can be also considered to be at a distance l_o from the centre of the circle.

(i) Find the Lagrangian of each system, assuming the 2 objects are moving in a gravitational field with acceleration g; and then find the equations of motion for $\theta(t)$, the angle of rotation about the circle centre, for the 2 systems.

(ii) Find the frequency of small oscillations for each of the 2 systems, about the "bottom" of the circle.

(iii) Now suppose that there is actually an internal frictional force inside the rotating solid cylinder, such that if the cylinder rotates about its own axis with angular velocity ω, the frictional force counteracting this rotation is $\eta\omega$. Find the new equation of motion for the angular coordinate $\theta(t)$ for the position of the cylinder centre; and solve this equation for small oscillations of the cylinder (with initial conditions left as free variables). What is the "Q-factor" for these oscillations (here Q is defined as the number of times the system oscillates before the oscillation amplitude falls by a factor e).

(iv) Finally - we can alleviate the problem of friction by attaching to the central solid cylinder, using massless spokes, an outer "wheel" of diameter $R_o \gg r_o$, fixed to rotate with the inner cylinder (but still with $R_o \ll l_o$); the system now rolls with the outer wheel in contact with the large circle. Let's assume that the mass of this wheel can be neglected. In this case the Q-factor will increase - what is its new value?

END of EXAM