QUICK NOTE ON FLUCTUATION DETERMINANTS

In the course notes a rather lengthy derivation is given of the fluctuation determinant for the path integral form of the Green function (section A.3.2(a)). However this derivation is, while fairly general, also rather unwieldy and is not very practical.

Here is a much simpler and more intuitively obvious method. Suppose we have a Lagrangian of form

$$L(\varphi,\dot\varphi;t) = \alpha(t)\dot\varphi^2 + \beta(t)\varphi\dot\varphi + \gamma(t)\varphi^2 + \eta(t)\dot\varphi + f(t)\varphi \qquad (1)$$

Now write the path $\varphi(t)$ as

$$\varphi(t) = \varphi_c(t) + q(t) \qquad (2)$$

where $\varphi_c(t)$ is the "classical path" that minimizes the action, and $q(t)$ is some small deviation about this path. We now wish to evaluate the action $S[\varphi]$ up to quadratic order in the deviations $q(t)$, $\dot q(t)$. We have

$$\left.\begin{aligned} S[\varphi,\dot\varphi] &= S[\varphi_c+q,\dot\varphi_c+q] \\ &= \int_{t_1}^{t_2} dt \Big\{ \alpha(t)\big[\dot\varphi_c^2 + 2\dot\varphi_c\dot q + \dot q^2\big] + \beta(t)\big[\varphi_c\dot\varphi_c + \varphi_c\dot q + q\dot\varphi_c + q\dot q\big] \\ &\qquad + \gamma(t)\big[\varphi_c^2 + 2\varphi_c q + q^2\big] + \eta(t)\big[\dot\varphi_c + \dot q\big] + f(t)\big[\varphi_c + q\big]\Big\} \end{aligned}\right\} \qquad (3)$$

from which all terms linear in q and $\dot q$ must vanish in the integral, since $S[\varphi_c,\dot\varphi_c]$ minimizes this integral - we thus get

$$S[\varphi,\dot\varphi] = S_c[\varphi_c,\dot\varphi_c] + \int_{t_1}^{t_2} dt\,\big[\alpha(t)\dot q^2 + \beta(t)q\dot q + \gamma(t)q^2\big]\Big|_0^0 \qquad (4)$$

where $\Big|_0^0$ indicates that

$$q(t_1) = q(t_2) = 0 \qquad (5)$$

by hypothesis (since $\delta\varphi(t_1) = \delta\varphi(t_2) = 0$ in the original variational problem).

Turning now to the path integral form for $G(\varphi_2,\varphi_1;t_2,t_1)$, we see that we can write

$$\left.\begin{aligned} G(\varphi_2,\varphi_1;t_2,t_1) &= \int_{\varphi_1(t_1)}^{\varphi_2(t_2)} \mathcal{D}\varphi(t)\, e^{i/\hbar\, S[\varphi,\dot\varphi]} \\ &\longrightarrow A(t_2-t_1)\, e^{i/\hbar\, S_c[\varphi_2,\varphi_1]} \end{aligned}\right\} \qquad (6)$$

where from (4):

$$A(t_2-t_1) = \int_{q(t_1)=0}^{q(t_2)=0} \mathcal{D}q(t)\, \exp\Big\{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\,\big[\alpha(t)\dot q^2 + \beta(t)q\dot q + \gamma(t)q^2\big]\Big\} \qquad (7)$$

We see that the result for $A(t)$ is completely independent of $\eta(t)$ and $f(t)$.

Consider now the example of a simple harmonic oscillator, with Lagrangian

$$L_0(\varphi,\dot\varphi) = \frac{1}{2}m(\dot\varphi^2 - \omega_0^2\varphi^2) \tag{8}$$

ie., we drop $\beta(t)$ in (1), and let $\alpha(t)$ and $\gamma(t)$ be constants. Then the fluctuation determinant $A(t_2-t_1)$ is just

$$A(t_2-t_1) \equiv A(T) = \int_0^0 \mathcal{D}q(t) \exp\left\{\frac{i}{\hbar}\int_0^T dt\, \frac{M}{2}(\dot q^2 - \omega_0^2 q^2)\right\} \tag{9}$$

where we define $T = t_2 - t_1$, noting that the result for A is translationally invariant in time.

To evaluate this, we use the condition that $q(0) = q(T) = 0$, and write $q(t)$ as a Fourier sum:

$$q(t) = \sum_{n=1}^{\infty} q_n \sin\left(\frac{\pi n t}{T}\right) \tag{10}$$

(there is no $n=0$ term - it would be zero). We then have

$$\left.\begin{aligned} \int_0^T dt\, q^2(t) &= \frac{T}{2}\sum_n q_n^2 \\ \int_0^T dt\, \dot q^2(t) &= \frac{\pi^2}{T^2}\sum_{n_1 n_2} n_1 n_2 \int_0^T dt \cos\left(\frac{n_1\pi t}{T}\right)\cos\left(\frac{n_2\pi t}{T}\right) = \frac{T}{2}\sum_n \left(\frac{n\pi}{T}\right)^2 q_n^2 \end{aligned}\right\} \tag{11}$$

so that

$$A(T) \propto \lim_{N\to\infty} \prod_{n=1}^{N} \int_{-\infty}^{\infty} dq_n \exp\left\{\frac{im}{2\hbar}\sum_n\left[\left(\frac{n\pi}{T}\right)^2 - \omega_0^2\right] q_n^2\right\} \tag{12}$$

where we have made the sum well-defined by truncating it to a finite number N, before taking the limit $N\to\infty$. We will not need to deal with the N-dependent normalisation factor here - this is why I simply write that the L.H.S. is proportional to the R.H.S. The integrals are simple Gaussians, so that

$$A(T) \propto \prod_{n=1}^{N}\left(\frac{n^2\pi^2}{T^2} - \omega_0^2\right)^{-1/2} \propto \left(\frac{\sin\omega_0 T}{\omega_0 T}\right)^{-1/2} \tag{13}$$

Now we can fix the constant of proportionality by noting that when $\omega_0 = 0$, we should get the easily computed free particle result:

$$A(T) \xrightarrow[\omega_0=0]{} \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} \tag{14}$$

Thus we have, for the SHO:

$$A(T) = \left(\frac{m\omega_0}{2\pi i\hbar \sin\omega_0 T}\right)^{1/2} \tag{15}$$