SLOW PROCESSES IN Q. MECHANICS

This is a topic of v. considerable subtlety, with many results only found in the last 10-20 years, and many problems still outstanding. The problem is inherently non-perturbative, and the results are of enormous theoretical importance, not just in ordinary QM; they form part of the whole underpinning of quantum field theory, both in its relativistic form & in its application to condensed matter systems.

We begin with a review of elementary results in the adiabatic limit, including the derivation of geometric phases, the Born-Oppenheimer approximation, and the Landau-Dykhne formula. We then continue with a discussion of the path integral formulation of these results, in preparation for a systematic treatment of deviations from the results in the adiabatic limit. This can be done in a systematic expansion in powers of a "slowness parameter", suggesting a renormalization procedure; but the expansion is intrinsically divergent and the non-perturbative character quickly becomes apparent. The problem raises quite interesting general questions in the theory of asymptotic expansions, as well as having important implications for the theory of effective Hamiltonians and effective Lagrangians.

1. ELEMENTARY RESULTS

The following results are considered to be basic to the whole subject. Even so, they are all rather subtle - thus, although elementary, they are by no means simple. We begin by recalling, without extensive derivation, the results of elementary time-dependent perturbation theory, and then consider the adiabatic limit of 1-particle QM. This leads to the idea of geometric phase, and the Born-Oppenheimer approx. We then review the basic "Landau-Dykne" theory of transitions in this limit, finishing with a brief geometric discussion of the whole picture in both Hamiltonian space and Hilbert space.

1.(a) TIME-DEPENDENT PERTURBATION THEORY

: This is a topic to be found in every book on Q.M.; however it contains some interesting features.

We begin with the time-dependent Schrödinger eqtn, in the form

$$H(t)\,\psi(t) = i\hbar\,\partial_t\,\psi(t) \qquad (1)$$

Now we can consider 2 obvious approximation schemes for this Hamiltonian. One obtains if the time-dependent part of $\mathcal{H}(t)$ is small; and the other applies if there is a clear separation of time-scales, ie., if the time-dependence is either very slow or very fast, compared to the characteristic timescales in the static part of the Hamiltonian. We look at these in turn.

(i) **WEAK PERTURBATION**: The simplest case, at first glance, is that of a weak perturbation. We write the Hamiltonian as

$$\mathcal{H} = \mathcal{H}_0 + V(t) \tag{2}$$

and assume

$$\hat{\mathcal{H}}_0 |\phi_n\rangle = \epsilon_n^0 |\phi_n\rangle \tag{3}$$

Then, in a sense to be specified, the perturbation $V(t)$ will be assumed small compared to $\mathcal{H}_0$. Thus we begin by writing things in terms of the ϕ_n, which satisfy the time-dependent equation

$$\hat{\mathcal{H}}_0 \phi_n(t) = i\hbar \partial_t \phi_n(t) \tag{4}$$

We assume a solution $\psi(t)$ of the full Schrodinger eqn (1) in the form of a sum over the ϕ_n:

$$\psi(t) = \sum_n a_n(t) \phi_n(t) \tag{5}$$

where

$$\phi_n(t) = \phi_n e^{-i/\hbar \epsilon_n^0 t} \tag{6}$$

Substituting (5) in (1), we find

$$\left.\begin{aligned} &\sum_n a_n(t)\left[(\mathcal{H}_0 - i\hbar\partial_t) + V(t)\right]\phi_n(t) - i\hbar \sum_n \dot{a}_n(t)\phi_n(t) \\ &= \sum_n \left[a_n(t) V(t) \phi_n(t) - i\hbar \dot{a}_n(t)\phi_n(t)\right] = 0 \end{aligned}\right\} \tag{7}$$

so, taking the inner product with $\phi_m^*(t)$, we have

$$\dot{a}_m(t) = -i/\hbar \sum_n V_{mn}(t) a_n(t) \tag{8}$$

This result is exact. From (6), we may also write it as

$$\dot{a}_n(t) = -i/\hbar \sum_m \bar{V}_{nm}(t) a_m(t) e^{i/\hbar(\epsilon_n^0 - \epsilon_m^0)t} \tag{9}$$

where $\bar{V}_{nm}(t) = \langle \phi_n | V(t) | \phi_m \rangle$. We note that $\bar{V}_{nm}(t)$ still depends on t, through the time dependence of $V(t)$ itself. Now

this equation is less simple than it looks at first glance - if the $\bar{V}_{nm}(t)$ are not small, we have a set of N coupled 1st-order differential equations, with t-dependent coefficients, and no hope of solution.

Accordingly we assume that $$V(t) \to \lambda V(t) \tag{10}$$

where in the 1st instance we assume $\lambda \ll 1$; this allows a perturbative expansion. We write

$$\left.\begin{aligned} a_n(t) &= \sum_{k=0}^{\infty} \lambda^k a_n^{(k)}(t) \\ &= a_n^{(0)} + \lambda a_n^{(1)}(t) + \lambda^2 a_n^{(2)}(t) + \cdots \end{aligned}\right\} \tag{11}$$

so that as $\lambda \to 0$, the system stays in one of the stationary states, i.e., $a_n(t) \to a_n^{(0)}$, a time-independent constant.

Equating powers of λ, after substituting (11) into (9), we find

$$\left.\begin{aligned} \dot{a}_n^{(k+1)}(t) &= -\tfrac{i}{\hbar} \sum_m a_m^{(k)}(t)\, V_{nm}(t) \\ &\equiv -\tfrac{i}{\hbar} \sum_m a_m^{(k)}(t)\, \bar{V}_{nm}(t)\, e^{\frac{i}{\hbar}(\epsilon_n^0 - \epsilon_m^0)t} \end{aligned}\right\} \tag{12}$$

Below we develop a systematic perturbative expansion in λ. First we recall the result for $\lambda \ll 1$, when it suffices to go to 1st order in λ. We then have

$$\dot{a}_n^{(1)}(t) = -\tfrac{i}{\hbar} \sum_m a_m^{(0)}\, \bar{V}_{nm}(t)\, e^{\frac{i}{\hbar}(\epsilon_n^0 - \epsilon_m^0)t} \tag{13}$$

from which we have

$$a_n^{(1)}(t) = -\frac{i}{\hbar} \sum_m a_m^{(0)} \int^t dt'\, \bar{V}_{nm}(t')\, e^{\frac{i}{\hbar}(\epsilon_n^0 - \epsilon_m^0)t'} \tag{14}$$

This eqtn., valid only for $\lambda \ll 1$, is often used to analyze problems in which an external AC field is treated classically (ie., as a scalar function $V(t)$).

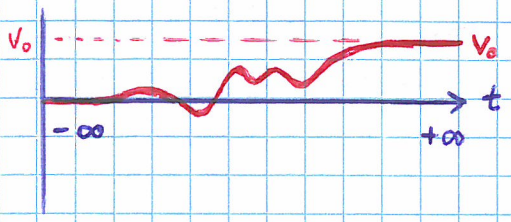
It is also employed in problems where $V(t)$ is switched on from zero to a finite value, so that

$$V(t) \xrightarrow[t \to -\infty]{} 0 \qquad\qquad V(t) \xrightarrow[t \to \infty]{} V_0 \tag{15}$$

In this case the integral in (14) is apparently undefined; however this pathology disappears if we integrate by parts:

$$a_n^{(1)}(t) = -\tfrac{i}{\hbar}\sum_m a_m^{(0)} \int_{-\infty}^{t} dt'\, \bar{V}_{nm}(t')\, e^{\frac{i}{\hbar}(\epsilon_n^0-\epsilon_m^0)t'}$$

$$= \sum_m \left\{ a_m^{(0)} \int_{-\infty}^{t} dt' \frac{\partial \bar{V}_{nm}(t')}{\partial t'} \frac{e^{\frac{i}{\hbar}(\epsilon_n^0-\epsilon_m^0)t'}}{\epsilon_n^0-\epsilon_m^0} - \left[\bar{V}_{nm}(t') \frac{e^{\frac{i}{\hbar}(\epsilon_n^0-\epsilon_m^0)t'}}{\epsilon_n^0-\epsilon_m^0} \right]_{-\infty}^{t} \right\} \qquad (16)$$



To understand the meaning of these terms, we note (see left) that the 2nd definite integral arises purely from the time-dependence of the shifted state when $t \to \infty$; recall we are assuming (15) at $t \to \infty$.

It is thus conventional to ignore the 2nd term when discussing the $t \to \infty$ limit, since it is simply equal to the time dependence of the state $\phi_n(t)$ coming from the shift in energy caused by switching on V_0; from time-INDEPENDENT perturbation theory we have

$$\phi_n = \sum_m{}' \phi_m \frac{V_{nm}^0}{\epsilon_n^0-\epsilon_m^0} \qquad (17)$$

so that the 2nd term in (16) is just the time-dependent contribution to $\phi_n(t)$ coming from the energy shift in (17). It is thus the 1st term in (16) that is describing real transitions. Thus one often writes the TRANSITION PROBABILITY in the $t \to \infty$ limit as

$$P_{nm} = \lim_{t\to\infty} |a_n^{(1)}(t)|^2 = \frac{1}{(\epsilon_n^0-\epsilon_m^0)^2} \left| \int_{-\infty}^{\infty} dt' \frac{\partial V_{nm}(t')}{\partial t'} e^{\frac{i}{\hbar}(\epsilon_n^0-\epsilon_m^0)t'} \right|^2 \qquad (18)$$

which makes clear that it is the time-varying part of the perturbation that is causing transitions.

(ii) PERTURBATION EXPANSION for GREEN FUNCTION : One can put the results in eqns (11) and (12) in a much more compact form by simply iterating to form a complete series expansion. Rather than writing this for the coefficients themselves, it is more sensible & compact to simply give

the series expansion for the propagator of the system. Accordingly we go back to the basic expression for the propagator of the system, written in operator form as

$$\hat{\mathcal{G}}(t,t') = \hat{G}^0(t,t') - \frac{i}{\hbar}\int_{t'}^{t} d\tau\, \hat{G}^0(t,\tau)\hat{V}(\tau)\hat{\mathcal{G}}(\tau,t') \qquad (19)$$

which when subdivided between the states $|\phi_n\rangle = |n\rangle$ and $|\phi_m\rangle$ is just

$$\mathcal{G}_{nm}(t,t') = G_n^0(t,t')\delta_{nm} - \frac{i}{\hbar}\int_{t'}^{t} d\tau\, G_n^0(t,\tau)V_{nm}(\tau)\mathcal{G}_m(\tau,t') \qquad (20)$$

Apart from this Dyson eqtn, it is also pedagogically useful to also consider the equivalent integral equation for the time evolution operator $U(t,t')$, which in the interaction representation takes the form

$$\left.\begin{aligned} \hat{U}(t,t') &= \hat{T}\exp\left\{-\frac{i}{\hbar}\int_{t'}^{t} d\tau\, \hat{V}(\tau)\right\} \\ \text{ie, which solves} \qquad \hat{V}(t)\hat{U}(t,t') &= i\hbar\partial_t U(t,t') \end{aligned}\right\} \qquad (21)$$

Both of these eqtns (19) and (21) can be developed as formal power series expansions. The time evolution operator is straightforward; the time-ordered expansion of (21) gives

$$\left.\begin{aligned} \hat{U}(t,t') &= 1 - \frac{i}{\hbar}\int d\tau\, \hat{V}(\tau)\hat{U}(\tau,t') \\ &= 1 - \frac{i}{\hbar}\int_{t'}^{t} d\tau_1\, V(\tau_1) + \left(\frac{i}{\hbar}\right)^2 \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2\, \hat{V}(\tau_1)\hat{V}(\tau_2) + \cdots \\ &\equiv 1 - \sum_{k=1}^{\infty}\left(\frac{-i}{\hbar}\right)^k \prod_{r=1}^{k}\int_{t'}^{\tau_{r+1}} d\tau_r\, \hat{V}(\tau_r) \end{aligned}\right\} \qquad (22)$$

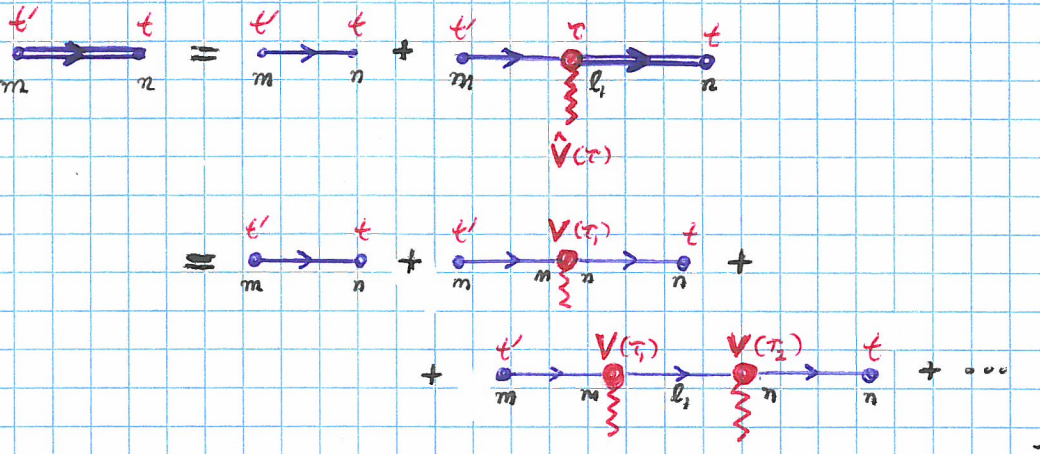
In the same way, the Green function can be expanded as

$$\left.\begin{aligned} \hat{\mathcal{G}}(t,t') &= \hat{G}_0(t,t') - \frac{i}{\hbar}\int_{t'}^{t} d\tau_1\, G^0(t,\tau)V(\tau)G^0(\tau,t') \\ &\quad + \left(\frac{-1}{\hbar}\right)^2 \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2\, G^0(t,\tau_1)V(\tau_1)G^0(\tau_1,\tau_2)V(\tau_2)G^0(\tau_2,t') \\ &\quad + \cdots \end{aligned}\right\} \qquad (23)$$

which is more unwieldy than (22) [the expansion (22) is so simple

just because we work in the interaction representation]. If we wish to write a compact expression for this series, we can do this as

$$\hat{G}(t,t') = G_0(t,t') + \sum_{k=1}^{\infty} \left(\frac{-i}{\hbar}\right)^k \prod_{r=1}^{k} \left[\int_{t'}^{\tau_{r+1}} d\tau_r \, \delta(t-\tau_0)\, G^0(\tau_{r-1}-\tau_r)\, V(\tau_r)\right] G^0(\tau_r, t') \tag{24}$$



(25)

If we sandwich these terms between eigenstates $|\Phi_m\rangle$ of $\mathcal{H}_0$, then we have

$$\mathcal{G}_{nm}(t,t') = G^0_{nm}(t,t') + \sum_{k=1}^{\infty} \left(\frac{-i}{\hbar}\right)^k \prod_{r=1}^{k} \left[\int_{t'}^{\tau_{r+1}} d\tau_r \sum_{\ell_r} \delta(t-\tau_0)\, \delta_{n\ell_0}\, G^0_{\ell_{r-1}}(\tau_{r-1}-\tau_r)\, V_{\ell_{r-1}\ell_r}(\tau_r)\right] \times G^0_m(\tau_r, t') \tag{26}$$

$$= G^0_{nm}(t,t') - \frac{i}{\hbar}\int_{t'}^{t} d\tau_1 \; G^0_n(t,\tau_1)\, V_{nm}(\tau_1)\, G^0_m(\tau_1,t') + \left(\frac{-i}{\hbar}\right)^2 \int d\tau_1 \int d\tau_2 \sum_{\ell_1} G^0_n(t,\tau_1) V_{n\ell_1}(\tau_1)\, G^0_{\ell_1}(\tau_1,\tau_2)\, V_{\ell_1}(\tau_2)\, G^0_m(\tau_2,t') + \cdots \tag{27}$$

We can make the sort of expansion in the eigenstate basis for (22). Notice that the lowest order (in $V(t)$) truncation of (27) just gives

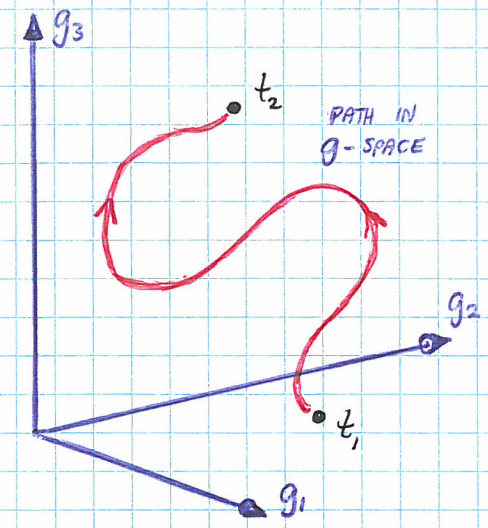
$$\mathcal{G}^{(1)}_{nm}(t,t') = G^0_{nm}(t,t') - \frac{i}{\hbar}\int_{t'}^{t} d\tau_1 \; G^0_n(t,\tau_1)\, V_{nm}(\tau_1)\, G_m(\tau_1,t') = G^0_{nm}(t,t') - \frac{i}{\hbar}\int_{t'}^{t} d\tau_1 \; V_{nm}(\tau_1)\, e^{i/\hbar(\epsilon^0_n - \epsilon^0_m)\tau_1} \tag{28}$$

which corresponds to the lowest-order result (14).

Formally, these expansions are still perturbative, and subject to all the limitations of perturbation expansions. In old-style perturbative developments of quantum field theory, equations (22) and (23) played a central role.

1.(b) ADIABATIC LIMIT & BERRY PHASE

Let us now consider a more general approach to the adiabatic limit. We begin by writing the Hamiltonian in the form



$$\left.\begin{aligned} \mathcal{H}(t) &= \mathcal{H}_0 + V(t) \\ &= \mathcal{H}(\underline{g}(t)) \end{aligned}\right\} \qquad (29)$$

where we parametrize the Hamiltonian now by a set of couplings

$$\underline{g}(t) = (g_1(t), g_2(t), \ldots g_N(t)) \qquad (30)$$

so that the Hamiltonian is now

$$\mathcal{H}(\underline{g}) = \mathcal{H}_0 + \sum_{n=1}^{N} g_n \hat{O}_n \qquad (31)$$

and where the $\hat{O}_n$ are operators acting in the Hilbert space of the system. Then, as we vary the time t, the system follows a path in "Hamiltonian space", ie, the space of couplings $\{g_n\}$. This is shown in the diagram above, in the case where $\underline{g}$ has 3 components. We imagine the system starting at $t = t_1$ and finishing at $t = t_2$.

(i) QM IN ADIABATIC REGIME: First consider the slow regime (ie., without yet going to the limit). We can then define, in the standard way, a set of adiabatic states as

$$\mathcal{H}(\underline{g})\, \phi_n(\underline{g}) = E_n(\underline{g})\, \phi_n(\underline{g}) \qquad (32)$$

The wave-function of the system at time t will then be

$$\psi(t) = \sum_n c_n(t)\, \phi_n(g(t))\, e^{-\frac{i}{\hbar}\int dt' E_n(g(t'))} \qquad (33)$$

and $\psi(t)$ obeys $\mathcal{H}(t)\psi(t) = i\hbar\, \partial_t \psi(t)$ (34)

If we now substitute (33) into (34), we have

$$\left.\begin{aligned} \mathcal{H}(t)\psi(t) &= \sum_n c_n(t) E_n(g(t)) \phi_n(g(t)) \, e^{-\frac{i}{\hbar}\int^t dt' E_n(g(t'))} \\ &= \sum_n \Big\{ i\hbar \big[\dot{c}_n(t) \phi_n(g(t)) + c_n(t) \partial_t \phi_n(g(t)) \big] \\ &\qquad + c_n(t) E_n(g(t)) \phi_n(g(t)) \Big\} \, e^{-\frac{i}{\hbar}\int^t dt' E_n(g(t'))} \end{aligned}\right\} \quad (35)$$

so that we have the interesting result that

$$\sum_n \big[\dot{c}_n(t) \phi_n + c_n(t) \partial_t \phi_n \big] e^{-\frac{i}{\hbar}\int dt' E_n(t')} = 0 \quad (36)$$

If we multiply this on the left by $\langle m(t)| = \phi_m^*(g(t))$, then we immediately find that

$$\left.\begin{aligned} \dot{c}_m(t) &= -\sum_n \langle m | \partial_t n \rangle c_n(t) \, e^{-\frac{i}{\hbar}\int^t dt' (E_n(t') - E_m(t'))} \\ &\equiv i \sum_n \underline{A}_{mn}(\underline{g}(t)) \cdot \dot{\underline{g}}(t) \; c_n(t) \, e^{-\frac{i}{\hbar}\int^t dt' (E_n(t') - E_m(t'))} \end{aligned}\right\} \quad (37)$$

for $m \neq n$, where we define a quantity

$$\left.\begin{aligned} \underline{A}_{mn}(\underline{g}) &= i \langle m | \partial_{\underline{g}} n \rangle \\ &\equiv i \langle \phi_m(\underline{g}) | \nabla_{\underline{g}} | \phi_n(\underline{g}) \rangle \end{aligned}\right\} \quad (38)$$

which we will see later can be thought of as a "gauge potential" in the space of Hamiltonians. For $m = n$, we get the simple result

$$\left.\begin{aligned} \dot{c}_n(t) &= -\langle n | \partial_t n \rangle \, c_n(t) \\ &= i \underline{A}_{nn}(\underline{g}) \cdot \dot{\underline{g}}(t) \; c_n(t) \end{aligned}\right\} \quad (39)$$

So far these results are quite general, and have nothing to do with the adiabatic regime. However we now define this limit a little more precisely. There are a number of ways of doing this; for example

(1) We define all timescales in terms of a characteristic time for the system in question. This timescale τ_0 will typically be the inverse (in units of $\hbar$) of a characteristic energy ϵ_0, which is often the level distance between the ground state energy and that of the first excited state (but this is

certainly not always the case). Then the adiabatic regime corresponds to

$$t\epsilon_0 \equiv t/\tau_0 >> 1 \qquad \text{(adiabatic regime)} \tag{40}$$

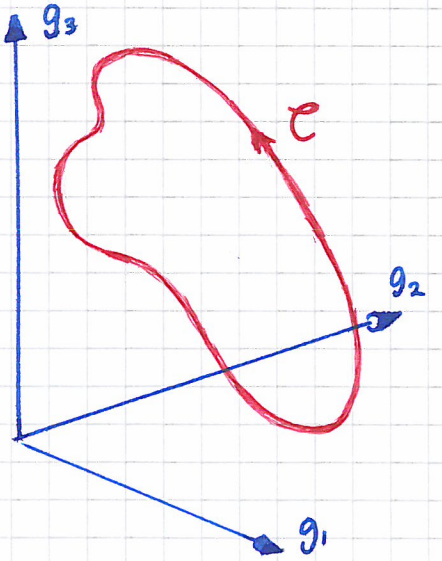
and the adiabatic limit is $t\epsilon_0 \to \infty$.

(ii) Another definition simply consists in defining a time scaling factor ε, which is used in all equations. Thus, we "stretch" time, by defining

$$\left.\begin{aligned} \varepsilon t &= \bar{\tau} \\ t &= \bar{\tau}/\varepsilon \end{aligned}\right\} \tag{41}$$

and the adiabatic regime is defined by $\varepsilon << 1$; the adiabatic limit is thus $\varepsilon \to 0$. This definition is less precise than the first one.

(iii) **BERRY PHASE & FLUX IN g-SPACE** : We now follow the interesting argument of Berry (which was anticipated by chemists like Longuet-Higgins, & Mead & Truhlar) and consider what happens when we transport a Hamiltonian adiabatically around a closed path in Hamiltonian space. The situation is as shown at left - we assume



$$\left.\begin{aligned} \mathcal{H}(t) &= \mathcal{H}(\underline{g}(t)) \\ \underline{g}(T) &= \underline{g}(0) \\ T &\to \infty \end{aligned}\right\} \tag{42}$$

The 1st of these eqtns makes the parameters $\underline{g} = (g_1 .. g_N)$ time-dependent; the 2nd eqtn. ensures we take the system through a closed circuit between times $t=0$ and $t=T$. The 3rd condition takes us to the adiabatic limit.

We define the instantaneous eigenstates as in (32). Then the "quantum adiabatic theorem" tells us that unless there is a level-crossing in the system, the system will, if it starts at $t=0$ in the state $|n\rangle \equiv |\phi_n\rangle$, then it must finish at $t=T$ in the same state $|n\rangle$. The meaning & derivation of this theorem can be found from eqtns. (37) & (39). If we let $\dot{\underline{g}}(t) \to 0$ in these 2 eqtns, then we go to the adiabatic limit. Let us now consider first the fate of state $|n\rangle$ during this circuit - this is the question addressed by Berry (in the next subsection we consider the time evolution of the states $|m\rangle$, for $m \neq n$).

From eqtns (33) and (39), we see immediately that the state of the system after a time t can be written as

$$\psi_n(t) = |n(\underline{g}(t))\rangle\, e^{-i\Phi_n(t)} \tag{43}$$

where

$$\Phi_n(t) = \phi_n(t) + \gamma_n(t) \tag{44}$$

with

$$\phi_n(t) = \frac{1}{\hbar}\int_0^t dt'\, E_n(\underline{g}(t')) \tag{45}$$

$$\left.\begin{aligned} \gamma_n(t) &= i\int_0^t dt'\, \dot{\underline{g}}(t')\langle n(g(t'))|\nabla_{\underline{g}}\, n(g(t'))\rangle \\ \text{ie}\quad \gamma_n(t) &= \int_{\underline{g}(0)}^{\underline{g}(t)} d\underline{g}\cdot \underline{A}_{nn}(\underline{g}) \quad\equiv\quad \int dt'\, \dot{\underline{g}}(t')\cdot \underline{A}_{nn}(g(t')) \end{aligned}\right\} \tag{46}$$

Now let's consider the result of taking the system through a circuit. We see that the following result obtains after a time t:

(a) The dynamical phase $\phi_n(T) = \frac{1}{\hbar}\int_0^T dt'\, E_n(\underline{g}(t'))$ accumulates. In a certain rather special sense we can think of this phase as a "clock" - it has measured the total time T (in units $\hbar/E_n(t')$, which we note vary around the circuit). Thus as $T\to\infty$, so does $\phi_n(T)$.

(b) The geometrical or "Berry" phase $\gamma_n(T)$, which is the integral of (46) around the circuit $\mathcal{C}$. To see its geometric significance, define the quantity

$$\underline{\mathcal{B}}_n(\underline{g}) = \nabla_{\underline{g}} \times \underline{A}_{nn}(\underline{g}) \tag{47}$$

which we can think of as an analogue to the EM field $\underline{B}(\underline{r})$ defined in real space. From (39) we see that $\underline{A}_{nn}(\underline{g})$ must be real (since $\langle n|\partial_t n\rangle$ must be imaginary; $\dot{c}_n$ is orthogonal to c_n). It then follows we can also write

$$\left.\begin{aligned} \underline{\mathcal{B}}_n(\underline{g}) &= g_m\left(\nabla_{\underline{g}} \times \langle n|\nabla_{\underline{g}}\, n(\underline{g})\rangle\right) \\ &= g_m \langle \nabla_{\underline{g}}\, n(\underline{g})| \times |\nabla_{\underline{g}}\, n(\underline{g})\rangle \end{aligned}\right\} \tag{48}$$

Now from (46) we clearly have

$$\gamma_n(T) = \oint_{\mathcal{C}} d\underline{g}\cdot \underline{A}_{nn}(\underline{g}) = \iint_{S_{\mathcal{C}}} d\underline{S}_{\underline{g}}\cdot \underline{\mathcal{B}}_n(\underline{g}) \tag{49}$$

where the curve $\mathcal{C}$ bounds the surface $S_{\mathcal{C}}$ in $\underline{g}$-space. The quantity $\mathcal{F}_n(\underline{g})$ thus represents a kind of field, and γ_n we now see represents a "flux" through this surface, ie, we have

$$\gamma_n(T) \xrightarrow[T\to\infty]{} \gamma_n(\mathcal{C}) \tag{50}$$

ie., $\gamma_n(\mathcal{C})$ depends only on the contour followed. Notice here that $\gamma_n(\mathcal{C})$ as a quantity does NOT depend on $\hbar$, ie., it is not necessarily even quantum-mechanical.

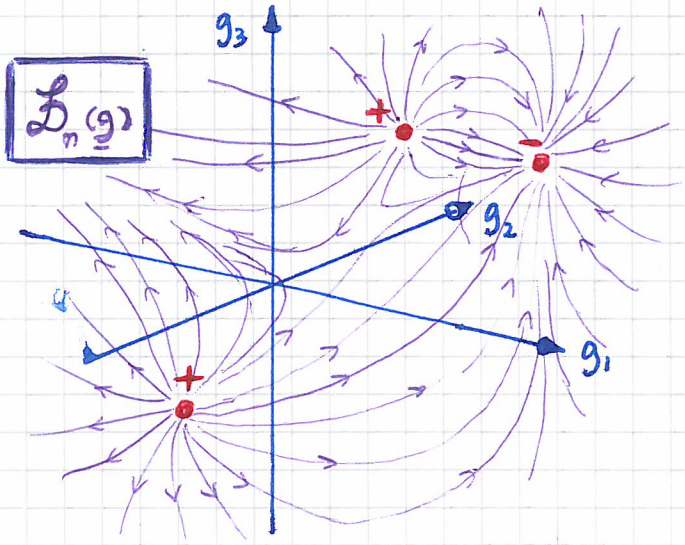
Obviously we would like to get a feeling for how the fields $\mathcal{A}_{nm}(\underline{g})$ and $\mathcal{F}_n(\underline{g})$ behave in $\underline{g}$-space. To do this, let us first note that for 2 states $|m(\underline{g})\rangle$ and $|n(\underline{g})\rangle$, the following identity is easily proved:

$$\langle m(\underline{g})|\nabla_{\underline{g}}\, n(\underline{g})\rangle = \frac{\langle m(\underline{g})|\nabla_{\underline{g}}\hat{\mathcal{H}}(\underline{g})|n(\underline{g})\rangle}{E_n(\underline{g}) - E_m(\underline{g})} \tag{51}$$

Then, by inserting intermediate states $|m\rangle$ into (48), we immediately get

$$\mathcal{F}_n(\underline{g}) = g_m \sum_{m\neq n} \frac{\langle n(\underline{g})|\nabla_{\underline{g}}\hat{\mathcal{H}}(\underline{g})|m(\underline{g})\rangle \times \langle m(\underline{g})|\nabla_{\underline{g}}\mathcal{H}(\underline{g})|n(\underline{g})\rangle}{\left(E_n(\underline{g}) - E_m(\underline{g})\right)^2} \tag{52}$$

This formula immediately makes it clear where the "sources" of the field $\mathcal{F}_n(\underline{g})$ lie in $\underline{g}$-space; they come from OTHER LEVELS - and in particular, from degeneracy with other levels. Typically this will only happen at points in $\underline{g}$-space (although one can in principle have higher-dimensional singular "degeneracy hypersurfaces". We thus have the typical (but not generic) situation shown in the picture, where the field issues from "degeneracy sources", which can have either sign, and which have a strength defined by (52). To see better how this works one has to consider examples.



Thus we see that the Hamiltonian or "$\underline{g}$-space" conceals a hidden geometry, populated by the analogue of a static gauge field - indeed, there is a whole set of

these fields, one for each of the levels corresponding to state $|n(\underline{g})\rangle$. The whole structure is very complex, because the different levels are connected together by these degeneracies, and so one has the generalization to N dimensions (the dimension of g-space) of a set of Riemann sheets.

If the number of levels that becomes degenerate exceeds two and/or if the degeneracy persists over a region of g-space, then it is simpler to rediscuss the whole problem in terms of a manifold of states. Thus, suppose the adiabatic states take the form:

$$\mathcal{H}(\underline{g})\,\phi_{n\nu}(\underline{g}) = E_n(\underline{g})\,\phi_{n\nu_n}(\underline{g}) \equiv E_n(\underline{g})\,|n\nu_n(\underline{g})\rangle \tag{53}$$

where $\nu_n = 1, \ldots p_n$ label a set of p_n different degenerate states, making up the manifold $\{\phi_{n\nu}(\underline{g})\}$ of these states in Hilbert space. The total wave function is then

$$\psi(t) = \sum_n \sum_{\nu_n} C_{n\nu_n}(t)\, e^{-\frac{i}{\hbar}\int^t dt' E_n(g(t'))}\, |n\nu_n(g(t))\rangle \tag{54}$$

and the coefficients $C_{n\nu_n}(t)$ obey the eqtn of motion

$$\left.\begin{aligned} \dot{C}_{m\alpha_m}(t) &= -\sum_n \sum_{\nu_n} \langle m\alpha_m | \partial_t | n\nu_n\rangle\, C_{n\nu_n}(t)\, e^{-i/\hbar \int^t dt' (E_n(g(t')) - E_m(g(t')))} \\ &\equiv i \sum_n \sum_{\nu_n} \underline{A}_{mn}^{\alpha_m \nu_n}(g(t)) \cdot \dot{\underline{g}}(t)\, C_{n\nu_n}(t)\, e^{-\frac{i}{\hbar}\int dt' (E_n(g(t')) - E_m(g(t')))} \end{aligned}\right\} \tag{55}$$

where $$\underline{A}_{mn}^{\alpha_m \nu_n}(\underline{g}) = i \langle m\alpha_m | \underline{\nabla}_g | n\nu_n\rangle \tag{56}$$

These eqtns are the generalizations of (32), (33), (37), & (38). Now let us specialize to the n-manifold, assuming these states do not intersect with (ie make transitions to) any other manifold. We then have, generalizing (39), that

$$\left.\begin{aligned} \dot{C}_{n\mu_n}(t) &= -\langle n\mu_n | \partial_t | n\nu_n\rangle\, C_{n\nu_n}(t) \\ &\equiv i\,\dot{\underline{g}} \cdot A_n^{\mu\nu}(\underline{g})\, C_{n\nu_n}(t) \end{aligned}\right\} \tag{57}$$

and the generalization of (43), the wave-function inside the n-manifold at time t, is

$$\psi_{n\mu}(t) = e^{i\Phi_n^{\mu\nu}(t)}\, |n\nu(g(t))\rangle \tag{58}$$

where the Berry and dynamical phases are now matrix quantities:

$$\underline{\Phi}_n^{\mu\nu}(t) = \phi_n^{\mu\nu}(t) + \gamma_n^{\mu\nu}(t) \tag{59}$$

with

$$\phi_n^{\mu\nu}(t) = \delta_{\mu\nu} \frac{1}{\hbar} \int dt' \, E_n(\underline{g}(t')) \tag{60}$$

$$\left.\begin{aligned} \gamma_n^{\mu\nu}(t) &= \int^t dt' \, \dot{\underline{g}} \cdot \underline{A}_{nn}^{\mu\nu}(\underline{g}(t')) \\ &\equiv \int_{\underline{g}(0)}^{\underline{g}(t)} d\underline{g} \cdot \underline{A}_{nn}^{\mu\nu}(\underline{g}) \end{aligned}\right\} \tag{61}$$

Thus we are dealing with a non-Abelian ($SU(p_n)$) generalisation of the original Berry phase. The resulting field is more complicated: one has

$$\underline{\mathcal{B}}_n^{\mu\nu}(\underline{g}) = \nabla_{\underline{g}} \times \underline{A}_{nn}^{\mu\nu}(\underline{g}) - i\left(\underline{A}_{nn}(\underline{g}) \times \underline{A}_{nn}(\underline{g})\right)^{\mu\nu} \tag{62}$$

with a corresponding increase in complexity of the field configurations. I will not pursue this interesting topic further here.

1(c) TRANSITIONS IN THE ADIABATIC REGIME

None of the above discussion took account of transitions between the different levels as one varied $\underline{g}(t)$; it was assumed that these were negligible (and they must be in the adiabatic limit - this is the content of the Q. adiabatic theorem).

However it is not obvious from eqtns. (34) or (55) what these transition amplitudes must be; nor is it even obvious that they go to zero in the adiabatic limit. A proper investigation of this question leads to some interesting mathematics, and a very useful formula, known as the "Landau-Dykhne" formula.

We consider the situation shown below, where the energy levels of some quantum system are plotted as a function of a single parameter g, a coupling in the Hamiltonian. Obviously we could imagine a set of hypersurfaces, like the energy surfaces $E_n(\underline{g})$; but we will not need these here, since we will only ever look at a particular path $\underline{g}(t)$ followed by the system over time t. Thus