# Notes:

- Graphs which did not display any useful information and just a “scribble” of points did not get full marks.
- For the “explain” parts of the marking scheme explanations that did not include references to the pendulum or the driving force were unlikely to get full marks.

# 1 Part 1:

The matlab script `Part1.m` solves the differential equation and creates the plots in this part of the project.
The equation of motion with the parameters given can be written as the first order system of equations

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\nu}{v} - \sin\theta + A\sin\omega t \tag{1.1}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = v. \tag{1.2}$$

The code solves equations (1.1-1.2) with $\nu = 1, 5, 10\mathrm{kgs}^{-1}$ , no driving, and the initial conditions $\theta = 0.2$ and $v = 0\mathrm{ms}^{-1}$. The solutions are shown in figure 1.1 which shows a plot of $\theta(t)$ and a phase portrait. In the $\nu = 1$ case the motion is *under-damped* and we see exponentially decaying oscillations of $\theta(t)$ as the initial energy is dissipated by friction. In this case in the phase portrait the coordinate spirals into the origin where the pendulum is at equilibrium. In the $\nu = 5$ and $\nu = 10$ case the motion is *over-damped* and we see no oscillations just an exponential decay in the $\theta(t)$ curve because energy is dissipated too fast for the pendulum to oscillate. The rate of the decay determined by $\nu$ so that the pendulum with $\nu = 10$ reaches equilibrium slower than the pendulum with $\nu = 5$. In the this case the phase portrait is shaped like a “tick” the the initial potential energy is converted to kinetic energy then dissipated, again the curve ends at the origin.

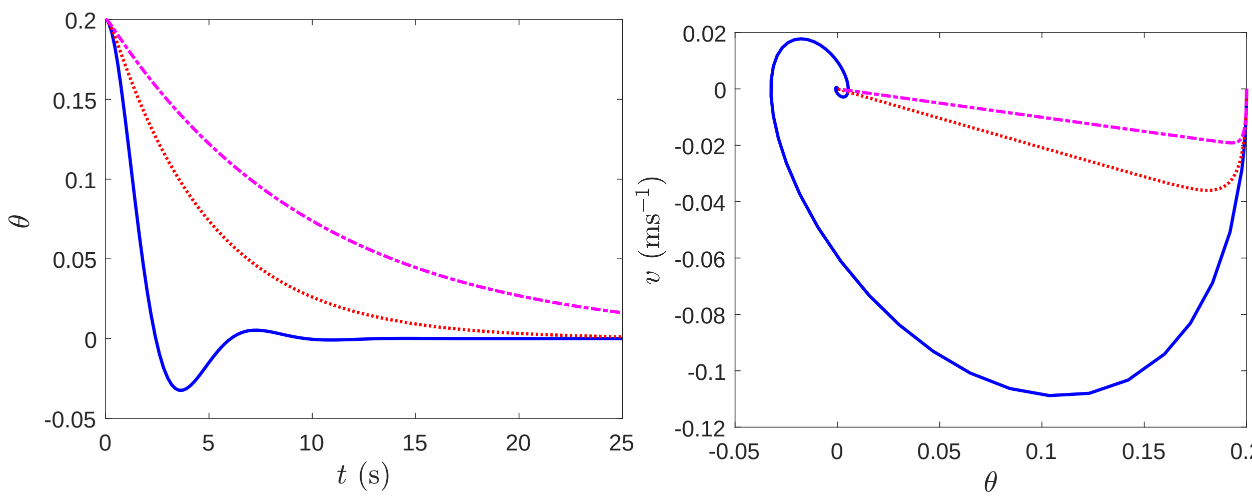


Figure 1.1: Plots of $\theta(t)$ (right) and the phase portrait (left) for the cases described in part one. Solid (blue) curves show results for $\nu = 1$ kgs$^{-1}$, the dotted (red) lines show results for $\nu = 5$ kgs$^{-1}$, and dash-dotted (magenta) lines show $\nu = 10$ results.

## 2 Part 2:

The matlab script `Part2.m` solves the differential equation and creates the plots in this part of the project.
The code solves equations (1.1-1.2) with $\nu = \frac{1}{2}$kgs$^{-1}$ , a driving force with frequency $\omega = \frac{2}{3}$ Hz,and the initial conditions $\theta = 0.2$ and $v = 0$ms$^{-1}$. The code solves for the motion for times $0 < t < 300T$, where $T = \frac{2\pi}{\omega} \approx 9.4$ s is the period of the driving force. Two different driving amplitudes are considered $A = 0.5$ N and $A = 1.2$ N. We present the results and discuss each of these cases separately in what follows. In both cases we use Matlab's inbuilt variable time step Runge-Kutta solver `ode45`. We use the following non-default settings for the solver for reasons discussed in part 3, the relative tolerance is set to $10^{-8}$ and the absolute tolerance is set to $10^{-10}$.

### $A = 0.5$ N

The results with $A = 0.5$ N are illustrated in figure 2.1. After initial transient behaviour the pendulum settles into periodic motion at the driving frequency. $\theta(t)$ for one period of this motion is shown in the left hand graph in figure 2.1 the curve looks approximately sinusoidal. The phase portrait for the first 300 periods of the forcing is shown in the right hand plot of figure 2.1 from which we see the coordinate spirals out from its initial point eventually ending in a closed orbit (in dynamical systems closed periodic motion is often referred to as a periodic orbit). This kind of motion is familiar the driving amplitude is small enough that we can treat pendulum as a driven damped harmonic oscillator (with small corrections due to the nonlinearities). In such an oscillator the motion of the oscillator tends to stable oscillation at the driving frequency with an amplitude determined by the balance of the driving force and dissipation.

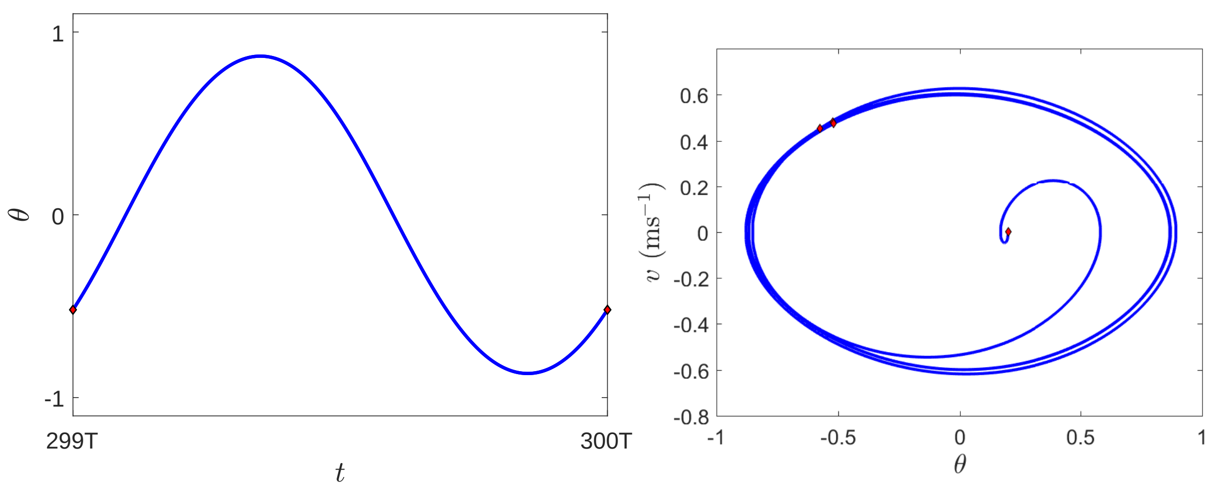


Figure 2.1: Plots of $\theta(t)$ (left) and the phase portrait (right) for the cases described in part the system with $A = 0.5$ N described in the part 2. On both plots as well as the curve (blue) we have included the Poincaré section (red diamond markers).

### $A = 1.2$ N

The results with $A = 0.5$ N are illustrated in figure 2.2. The motion does not settle to any kind of stable orbit. The left hand graph of figure 2.2 shows a plot of $\theta(t)$ for this case I have chosen not to map the $\theta$ variable on to the range $-\pi < \theta < \pi$ to illustrate the structure of the motion. The diving amplitude is such that $\theta$ is no longer "small" in fact the pendulum can now be vertical (when $\theta = n\pi$ for $n$ odd) and the driving period is incommensurate with the time it typically takes for the pendulum to do one rotation so sometimes when the pendulum is at the top it tips over to the left and other times it tips over to the right. Which way the pendulum falls when it is at the top depends what the force is at that time and the velocity of the pendulum. Sometimes the pendulum tips over and swing multiple times around its centre and other times it does not. Because of this it appears that whether pendulum swings one way or the other is random. The pendulums motion in this case is *chaotic*. The phase portrait (figure 2.2 on the right) reveals some of complicated structure of the motion, it is bounded but does not tend to a fixed line instead it fills out some region of phase space and is more densely packed in certain regions.

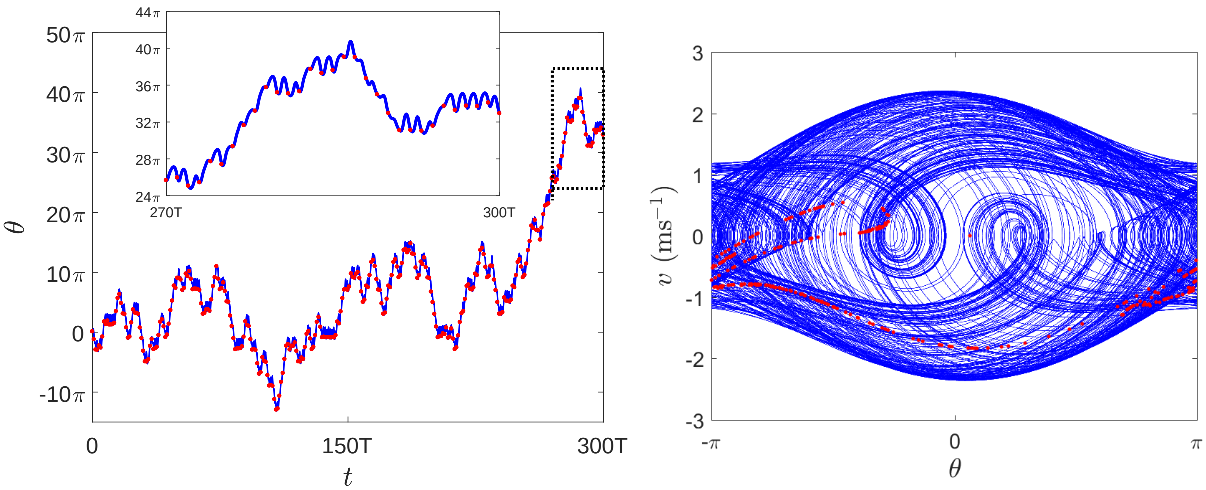


Figure 2.2: Plots of $\theta(t)$ (left and inset) and the phase portrait (right) for the cases described in part the system with $A = 1.2$ N described in the part 2. On both plots as well as the curve (blue) we have included the Poincaré section (red markers). The inset in the $\theta(t)$ shows an enlargement of the area in the dotted black rectangle.

# 3 Part 3:

The code for this section is contained in the Matlab file `Part3.m`, the code produces a lot of graphs, takes a long time to run and is quite messy. This reflects "trail and error" approach that one must adopt when checking these kind of convergence issues. The code produces the graphs in this section and similar graphs for all cases considered in part 2 and 4.

The goal of this section is to ensure that that the results we have reflect the actual solutions to equations (1.1 - 1.2) and are not just an artifact of our numerical procedure. In practice we need to chose values for the relevant solver parameters that are small enough so that making them smaller will not effect our results. It is possible to make these too small so that machine precision becomes an issue so one needs to be careful.
In the solutions I have used the solver `ode45` which has two parameters that are important for this section, the relative tolerance (`RelTol` in matlab) and the absolute tolerance (`RelTol` in matlab)[1]. It makes sense to have an absolute tolerance which is smaller than the relative tolerance (why?) so I set `AbsTol`=`RelTol`/100 and change `RelTol`.
We have seen above that there are two important cases for the driven damped pendulum, periodic motion and chaotic motion. It turns out that the way the solver converges to these types of solution is different for each of these cases so we will treat them separately. I stress that it is important to be careful when characterising motion as chaotic as insufficiently small solver parameters can make periodic motion look chaotic.

## Periodic motion.

The $A = 0.5$ N case studied in part 2 and all of the cases in part 4 result in periodic motion. The first way I have checked for the convergance of the solution was by plotting result calculated with different solver parameters on top of each other and checking the curves have converged. One such plot for the $A = 0.5$ case is shown in figure 3.1 where we have plot $\theta(t)$ with different tolerances. From this plot we see that the curves for `RelTol`$= 10^{-4}$ and `RelTol`$= 10^{-6}$ produce an almost identical plot. We are also interested in the behaviour of the phase portrait so we should look the effect of changing the tolerance on the phase portrait. In figure 3.2 we have the phase portrait for the $A = 0.5$ N with time values $150T < t < 300T$ with different tolerances. This plot shows that when tolerance is too large (`RelTol`$= 10^{-2}, 10^{-4}$) the orbit which would other wise be a closed oval is "smeared out" as the lack of accuracy causes the orbit to not quite close. A similar phase portrait is shown in figure 3.3 in which the $A = 1.465$ solution studied in part 4 shown with different tolerances. We see that in this case the smearing is worse.

[1]One could argue that `hmax`is also important but the default value for this is fine in most cases including those in this project.

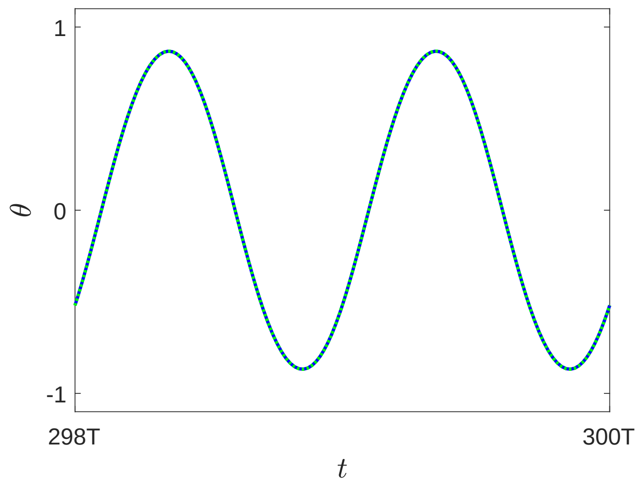


Figure 3.1: Eyeballing the convergence of the $A = 0.5$N case. The solid (blue) line shows the last few periods of the curve calculated with a relative tolerance of $10^{-4}$ and an absolute tolerance of $10^{-6}$. The dotted (green) line shows the last few periods of the curve calculated with a relative tolerance of $10^{-6}$ and an absolute tolerance of $10^{-8}$.

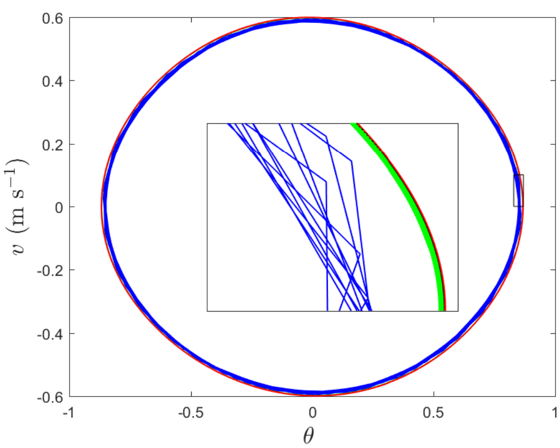


Figure 3.2: Eyeballing the convergence of the $A = 0.5$N case with the phase portrait. The solid (blue) line shows the phase portrait for $150T < t < 300T$ calculated with a relative (absolute) tolerance of $10^{-2}$ ($10^{-4}$), the green curve $10^{-4}$ ($10^{-6}$), the red curve $10^{-6}$ ($10^{-8}$), and the black curve has tolerances of $10^{-8}$ ($10^{-10}$). Inset is a magnification of the boxed area.

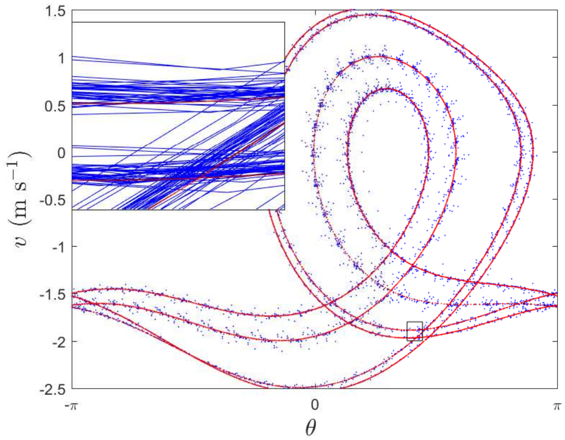


Figure 3.3: Eyeballing the convergence of the $A = 1.456$N case described in part 4 with the phase portrait. The blue points show the phase portrait for $150T < t < 300T$ calculated with a relative (absolute) tolerance of $10^{-4}$ ($10^{-6}$), the black points $10^{-6}$ ($10^{-8}$), and the red points are calculated with the tolerances $10^{-8}$ ($10^{-10}$).

Now that we have seen how our plots converge it makes sense to get an idea of how precise our solutions are. Figure 3.4 shows the absolute value of differences between successive numerical approximations to $\theta(t)$ damped driven pendulum for the $A = 0.5$ N case described in part 2. We see that across the whole time range studied the difference between these numerical approximations decreases regularly as the tolerance is decreased. Because of this we *can expect* to be able to calculate the angle of the pendulum with some accuracy after 300 periods of forcing in this case.

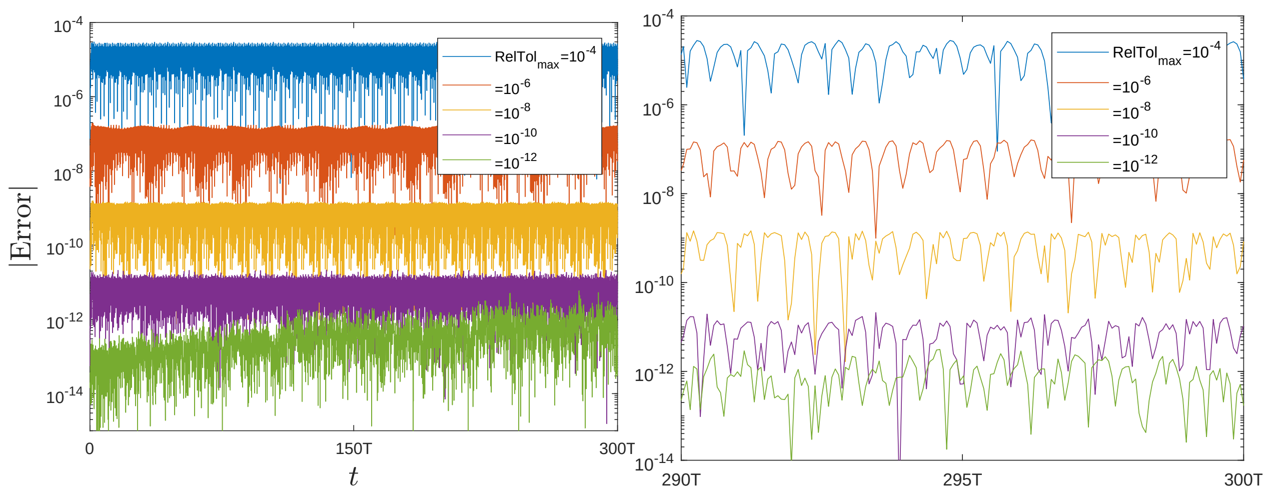


Figure 3.4: Plots of the absolute value of the difference between successive numerical approximations to $\theta(t)$ damped driven pendulum for the $A = 0.5$ N case described in part 2. The left hand graph shows the error evaluated over the full 300 period of the driving force and the right hand plot shows only the last 10 period. Curves are labeled in the legend by the maximum `RelTol` parameter used in each case (the `AbsTol` in each case is `AbsTol`=`RelTol`/100 ). E.g. the blue curve is $|\theta_{\texttt{AbsTol}=10^{-4}} - \theta_{\texttt{AbsTol}=10^{-6}}|$. Inset is a zoomed in plot of the boxed region with the points connected by lines.

## Chaotic motion.

When we compare the results calculated with different tolerances for chaotic motion we see that convergence is not as fast as in the case of periodic motion. Figure 3.5 shows us $\theta(t)$ calculated with different tolerances in the $A = 1.2$

N case studied in part 2. We see that curves with bigger tolerances diverge from those with smaller tolerances earlier in the motion of the pendulum and that decreasing the tolerance by a couple of *orders of magnitude* only changes increases the time for which the curves agree by a couple of periods of the forcing. One of the characteristics of a chaotic system is that it amplifies small changes exponentially over time so that small errors (for example rounding errors) become large enough completely change the motion of the pendulum after a relatively small number of periods of the forcing. This sensitive dependence on small changes is what is causing the problems we are seeing with the convergence. We can also see the symptoms of this sensitive dependence in a plot of the successive differences shown (for the same situation) in figure 3.6. These figures show us that we *cannot expect* to predict the location of the chaotic driven damped pendulum after 300 forcing periods.

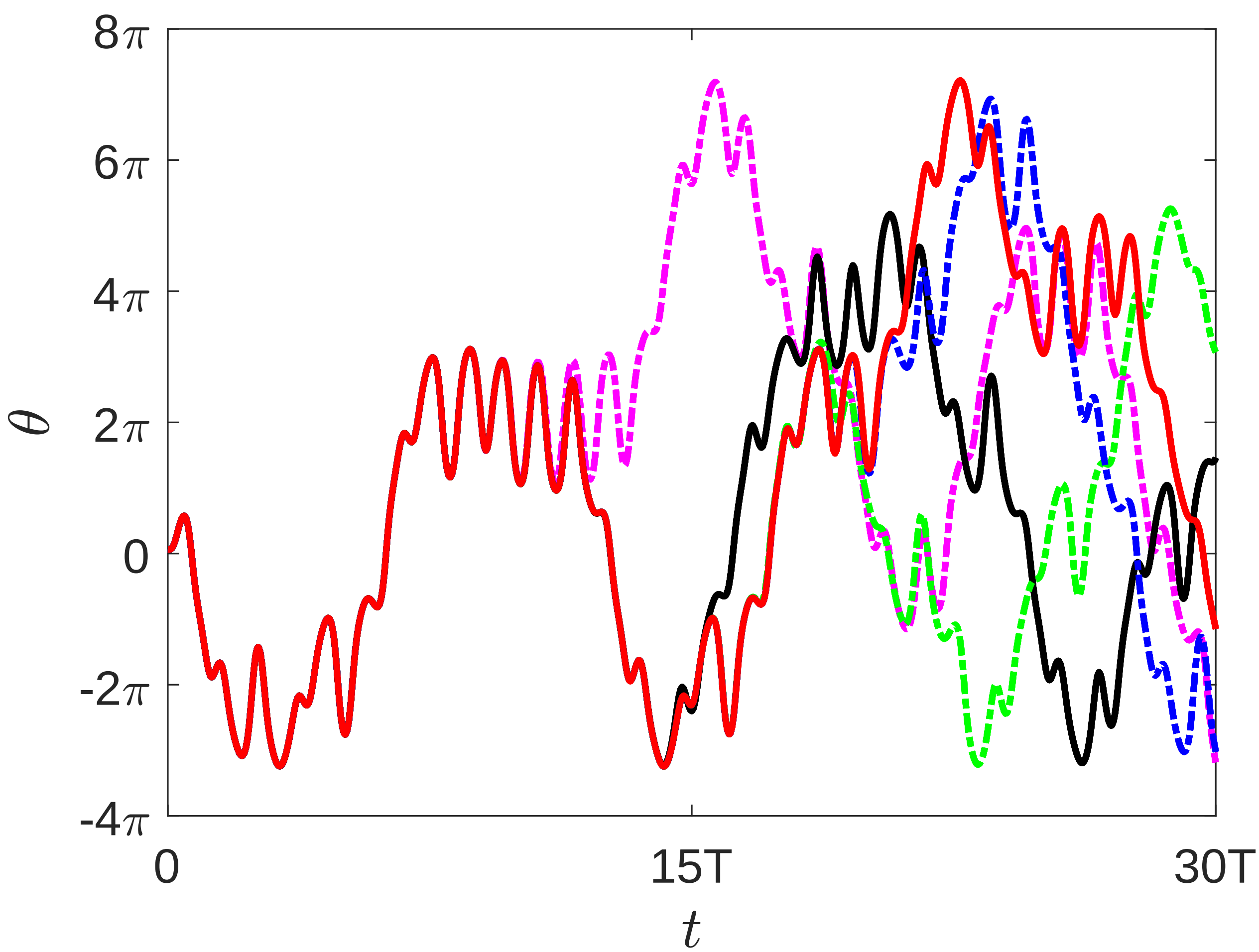


Figure 3.5: Eyeballing the convergence of the $A = 1.2$N case. The curves have the following absolute (relative) tolerances: dotted magenta $10^{-8}$ ($10^{-10}$), solid black $10^{-10}$ ($10^{-12}$), dashed green $10^{-12}$ ($10^{-14}$), dashed blue $10^{-14}$ ($10^{-16}$), and solid red $10^{-16}$ ($10^{-18}$)

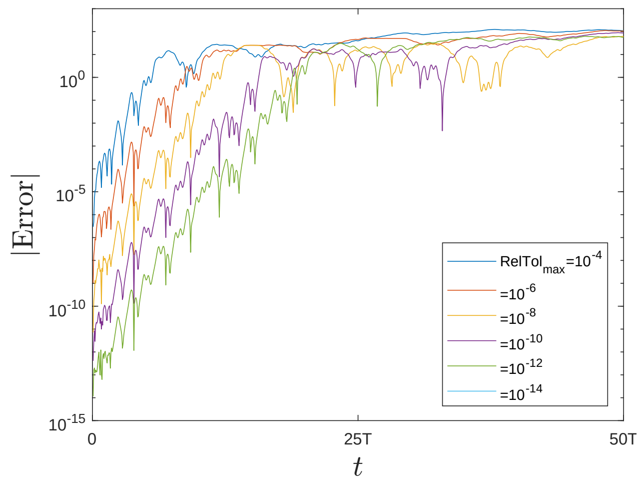


Figure 3.6: Plots of the absolute value of the difference between successive numerical approximations to the motion of the damped driven pendulum for the $A = 1.2$ N case described in part 2. The left hand graph shows the error evaluated over the full 300 period of the driving force and the right hand plot shows only the last 10 period. Curves are labeled in the legend by the maximum `RelTol` parameter used in each case (the `AbsTol` in each case is `AbsTol`=`RelTol`/100 ). E.g. the blue curve is $|\theta_{\texttt{AbsTol}=10^{-4}} - \theta_{\texttt{AbsTol}=10^{-6}}|$.

When we look at how the phase portrait changes when we decrease the tolerance we see that we can predict qualitative features of the pendulums motion even if we can't predict the pendulum's position far in the future. Figure 3.7 shows phase portraits for the $A = 1.2$ N system from part 1 with different tolerances and we see that while the exact trajectory of the pendulum through phase space is not the same the phase portrait and the interesting structure are qualitatively unchanged once the tolerance is low enough.

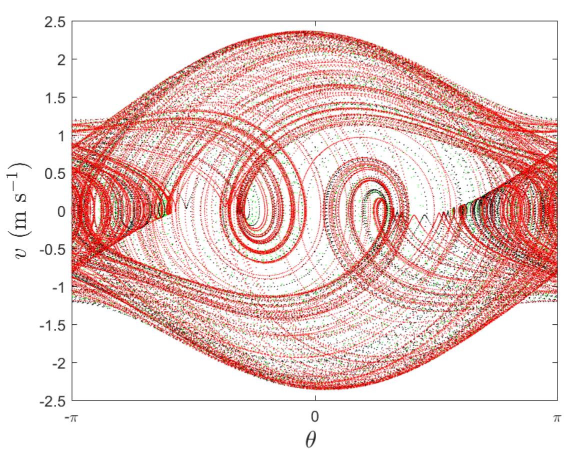


Figure 3.7: Eyeballing the convergence of the $A = 1.2$N case with the phase portrait. The green points show the phase portrait for $150T < t < 300T$ calculated with a relative (absolute) tolerance of $10^{-4}$ ($10^{-6}$), the black points $10^{-6}$ ($10^{-8}$), and the red points $10^{-8}$ ($10^{-10}$).

## Parameters used in the rest of the solutions

Just to be safe I used a relative (absolute) tolerances of $10^{-8}$ ($10^{-10}$) for the solutions to parts 2,3,5, and 6 based on the above. This is probably overkill, relative (absolute) tolerances of $10^{-6}$ ($10^{-8}$) are definitely sufficient.

# 4 Part 4:

The code solves equations (1.1-1.2) with $\nu = \frac{1}{2}\mathrm{kgs}^{-1}$ , a driving force with frequency $\omega = \frac{2}{3}$ Hz, and the initial conditions $\theta = 0.2$ and $v = 0\mathrm{ms}^{-1}$. The code solvers for the motion over 300 of the period $T = \frac{2\pi}{\omega} \approx 9.4$s of the driving force . Three different driving amplitudes are considered $A = 1.35$ N, $A = 1.44$ N, and $A = 1.465$ N. We present the results and discuss each of these cases separately in what follows. In all cases we use Matlab's inbuilt variable time step Runge-Kutta solver `ode45`. We use the following non-default settings for the solver for reasons discussed in part 3, the relative tolerance is set to $10^{-8}$ and the absolute tolerance is set to $10^{-10}$.

## $A = 1.35$ N

The results with $A = 1.35$ N are illustrated in figure 4.1. After initial transient behaviour the pendulum settles into periodic motion at the driving frequency. $\theta(t)$ for one period of this motion is shown in the left hand graph in figure 4.1 we see that every period the the pendulum completes one clockwise rotation about the centre and each period contains an interval of time where the pendulum is rotating in the anti-clockwise direction. The phase portrait for the first 300 periods of the forcing is shown in the right hand plot of figure 4.1 from which we see that the final period orbit of the pendulum in phase space has loop (in addition to winding around the an angle of $2\pi$). We can conclude from these figures that the driving amplitude and initial energy are such that in one period of the driving frequency the force is large enough to propel the pendulum all the way around even with dissipation.

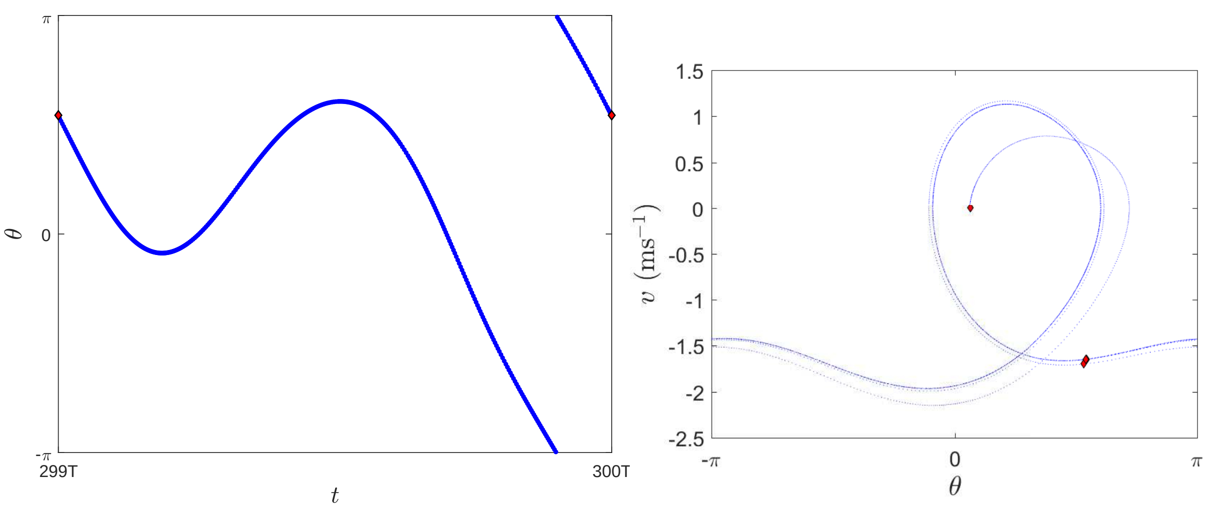


Figure 4.1: Plots of $\theta(t)$ (left) and the phase portrait (right) for the cases described in part 4 with $A = 1.35$ N. On both plots as well as the full plot (blue) we have included the Poincaré section (red diamond markers).

## $A = 1.44$ N

The results with $A = 1.44$ N are illustrated in figure 4.2. After initial transient behaviour the pendulum settles into periodic motion at the *half* of the driving frequency. A plot of $\theta(t)$ for one period of the pendulum's motion is shown in the left hand graph in figure 4.2 we see that every period the the pendulum completes two clockwise rotations about the centre and each period contains two distinct intervals of time where the pendulum is rotating in the anti-clockwise direction. Comparing the two $\theta(t)$ graphs figures 4.1 and 4.2 we can see that *period-doubling* has occurred, one period of the pendulums motion with $A = 1.44$ N takes corresponds to two periods of the pendulums motion with $A = 1.35$ N. The difference between the two cases being that the even numbered reversals of the pendulums velocity are longer than than the odd numbered reversals. There is another possible stable orbit (corresponding to different initial conditions) where this is the opposite way around so we say the stable periodic orbit seen in the $A = 1.33$N case has *bifurcated* into two orbits. The phase portrait for the first 300 periods of the forcing is shown in the right hand plot of figure 4.2 from which we see that the final period orbit of the pendulum in phase space has two distinct loops (in addition to winding around the an angle of $4\pi$). A possible interpretation of these results is that the driving amplitude now gives the pendulum a little more than enough energy that would see it winding all the way around in one period of the driving and that the in the stable orbit ever second driving period is different which compensates for this extra energy.

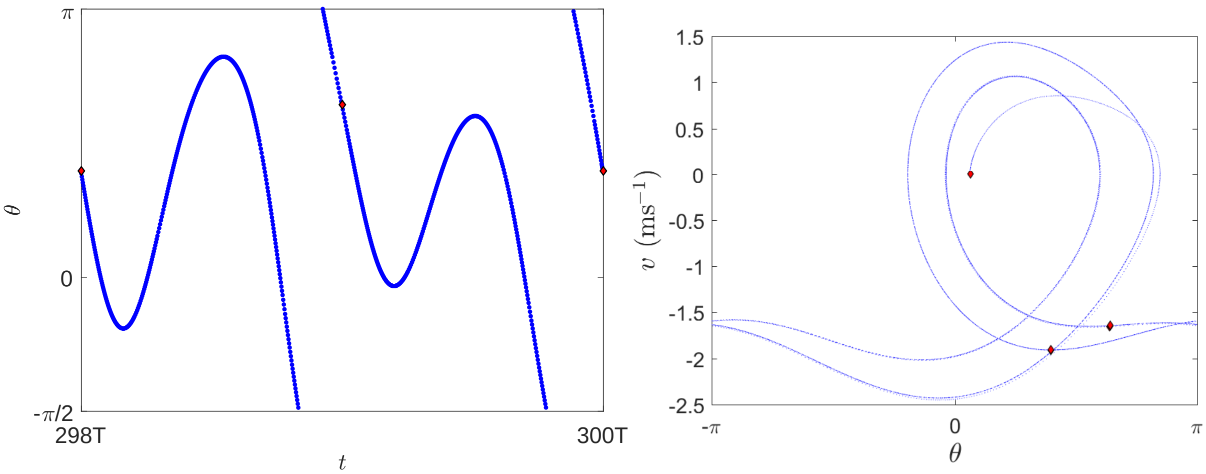


Figure 4.2: Plots of $\theta(t)$ (left) and the phase portrait (right) for the case described in part 4 with $A = 1.44$ N. On both plots as well as the full plot (blue) we have included the Poincaré section (red diamond markers).

## $A = 1.465$ **N**

The results with $A = 1.465$ N are illustrated in figure 4.2. After initial transient motion the pendulum settles into periodic motion at the *quarter* of the driving frequency. One period of the pendulum's motion is shown in the top left plot in figure 4.3 we see that another period doubling bifurcation has occurred. Every period the the pendulum completes four clockwise rotations about the centre and each period contains four distinct intervals of time where the pendulum is rotating in the anti-clockwise direction. In the phase portrait for the first 300 periods of the forcing (shown in the top right hand plot of figure 4.3) the final perodic orbit is a a obscured by the transient motion so I have included a plot including only the later times $150T < t < 300T$ (bottom of figure 4.3). There is a similar interpretation to the motion in this case as in the $A = 1.44$ N motion.

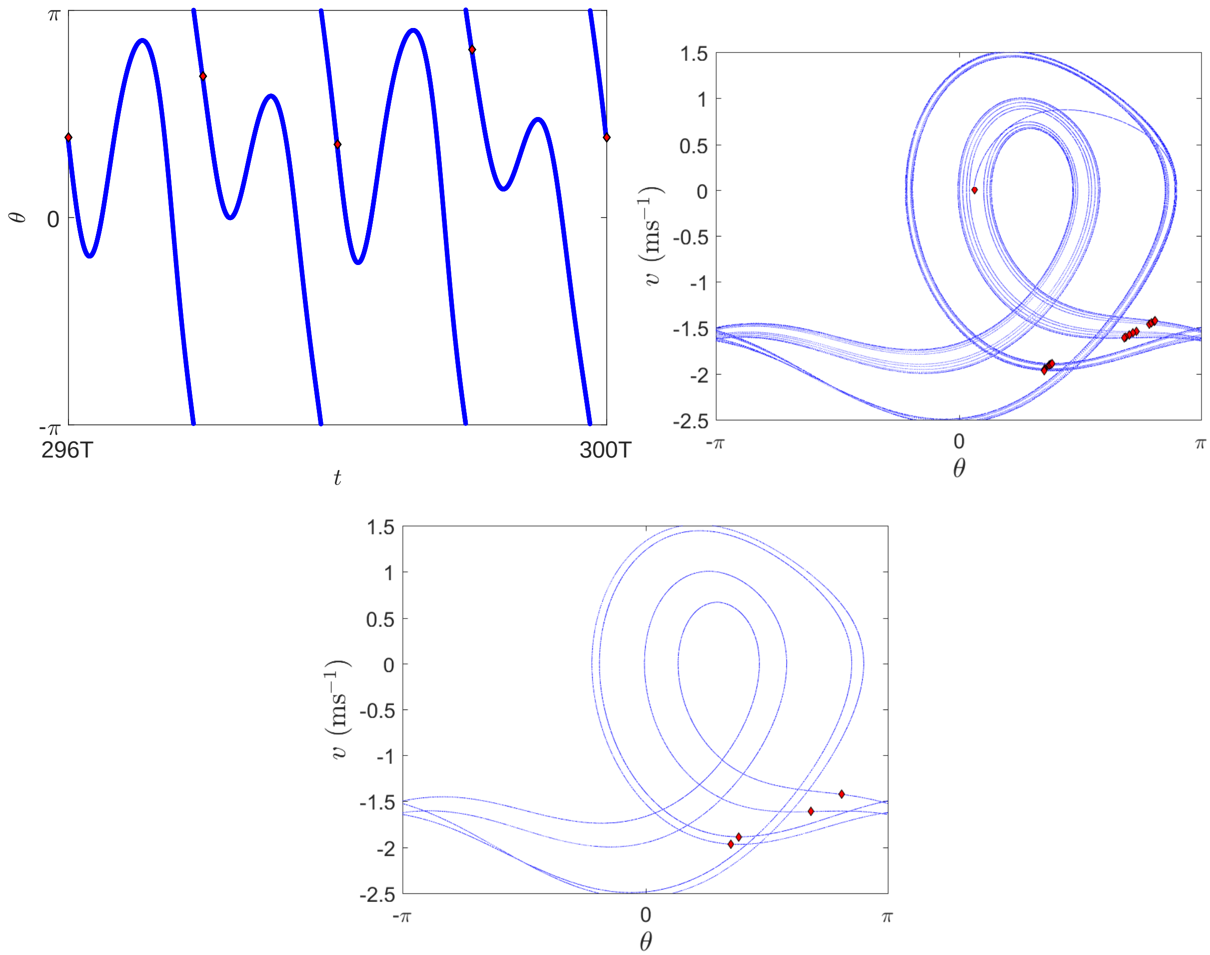


Figure 4.3: Plots of $\theta(t)$ (top left) with $0 < t < 300T$, the phase portrait (top right) with $0 < t < 300T$, and the phase portrait with $150T < t < 300T$ (bottom) for the case described in part 4 with $A = 1.465$ N . On all plots as well as the curve (blue) we have included the Poincaré section (red diamond markers).

# 5 Part 5:

Most of the work in this section is done by the codes for parts 2 and 4 (the scripts are `Part2.m` and `Part4.m` respectively). The one extra graph in this section is generated by `Part5.m`.

Poincaré sections asked for in the question are shown on the figures 2.1, 2.2,4.1,4.2, and 4.3 in parts 2 and 4. We interpret these plots in this section then describe the physical significance.
In the phase space Poincaré plots for the $A = 0.5$ N and $A = 1.35$ N shown in the right hand graphs of figures 2.1 and 4.1 respectively tend to a single point after a number of periods. The origin of this convergence is clear if we look at the $\theta(t)$ plots (left hand graphs on figures 2.1 and 4.1) the pendulum's motion tends to periodic motion with the same period as the driving force.
We see slightly different behaviour the phase space Poincaré plots for the $A = 1.44$ N and $A = 1.465$ N shown in the right hand graphs of figures 4.2 and 4.3 respectively. These Poincaré plots tend to two distinct points and four distinct points respectively. Again we can see from the $\theta(t)$ Poncaré plots 4.2 and 4.3 that the doubling of the points in the Poincaré phase portrait comes from the period doubling of the pendulums motion.
In the chaotic case with $A = 1.2$ N $\theta(t)$ Poncaré plot shown in figure 5.1 appears random because of the chaotic motion of the pendulum. In the Poincaré phase portrait shown in the right hand graph of figure 2.2 reveals that there is some structure the points all lying in a specific region of phase space. This is investigated further in the script `part5.m`

which produces a phase space Poincaré plot with more points (I include $10^5$ periods after the first 300) at later times (I have excluded all the points with $t < 300T$). The plot produced is shown in figure 5.2 and we see that the points in the long time Poincaré section lie in a complicated region that has a fractal structure.

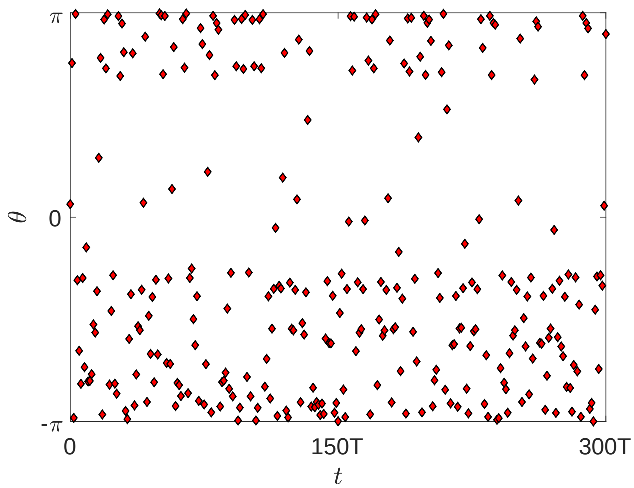


Figure 5.1: The Poincaré section of $\theta(t)$ when $A = 1.2$ N.

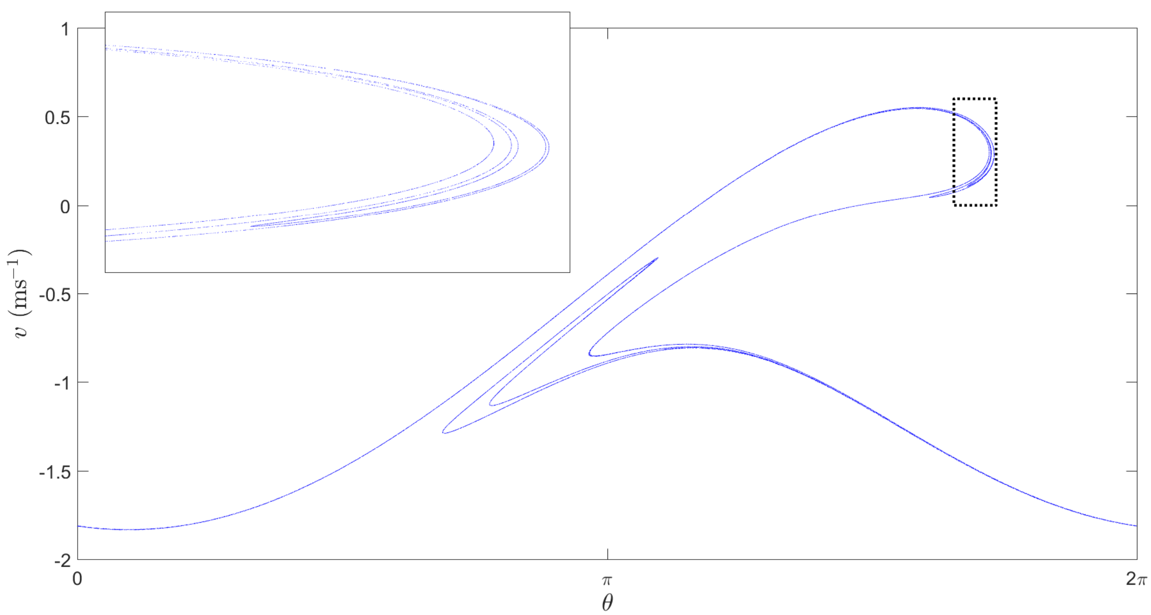


Figure 5.2: The Poincaré section of the phase portrait when $A = 1.2$ N. Inset graph an enlargement of the region inside dotted box. Here the angle is taken to be between $0$ and $2\pi$ so that the graph appears centred.

We can understand the physics of what is happening here as follows (this is mostly a combines the observations made in the discussions in previous sections). $A = 0.5$ N is a low enough driving amplitude that the in the long time limit the pendulum undergoes periodic motion something like that of a driven damped simple harmonic oscillator at the frequency of the driving force. At $A = 1.2$ N the driving amplitude is enough to tip the pendulum over the top in some cases and the motion is chaotic as the frequency of the driving is not right for the pendulum to set into period motion with this amplitude of driving. Once the amplitude of the driving force is increased to $A = 1.35$ N the pendulum the amplitude and frequency of the driving allow for stable periodic motion of the pendulum again only this time the pendulum completes a full revolution each driving period. When the driving amplitude is increased further to $A = 1.44$ N a period doubling bifurcation occurs and in order to have damp out energy at the same rate that the driving is providing it the pendulum needs to have different motion on alternating cycles of the driving. Increasing the amplitude further to $A = 1.465$ N causes another period doubling bifurcation and the stable oscillation of the pendulum has a period which takes as long as four driving periods.

# 6 Part 6:

Scripts used to generate the graphs in this section are `Part6.m` and `Part6b.m`.

The code investigates the long time Poincaré section of $\theta(t)$ described by the equations of motion (1.1-1.2) with $\nu = \frac{1}{2}\text{kgs}^{-1}$ , a driving force with frequency $\omega = \frac{2}{3}$ Hz, and the initial conditions $\theta = 0.2$ and $v = 0\text{ms}^{-1}$. Two ranges of driving amplitude are investigated $0.5$ N$< A < 1.2$ N and $1.35$ N$< A < 1.5$ N. We describe the results for amplitudes in the range $1.35$ N$< A < 1.5$ N first as these are simpler to interpret (full marks for the description were given for explaining one of the figures) .

## $1.35$ N$< A < 1.5$ N

The script `Part6b.m` relates to this subsection. The code takes $\sim 30$ minutes to run.

The code calculates 50 Poincaré points $\theta(Tn)$ are after the pendulum has be in motion for 300 driving periods for different values of the driving amplitude. Plots of the Poincaré points vs the driving amplitude are shown in figures 6.1, 6.2, and 6.3. We can see as we increase the driving amplitude that preceding the transition to chaos there are a series of period doubling bifurcations, until about $A \approx 1.48$ N (see the enlargement in figure 6.2) where there are complicated orbits for which the Poincaré angle values appear to fill two regions of possible $\theta$ values (we would have to do more work to see if this was chaotic motion or not). The orbits then coalesce back to an orbit with twice the period of the driving force followed by an apparently abrupt transition to chaos at $A \approx 1.4917$ N. Increasing $A$ further (see enlargement in figure 6.3) we then see there is a "window of order" for $1.492$ N$\lesssim A \lesssim 1.495$ N where the orbits are again periodic with period doubling bifurcations occurring followed by a full transition to chaos.

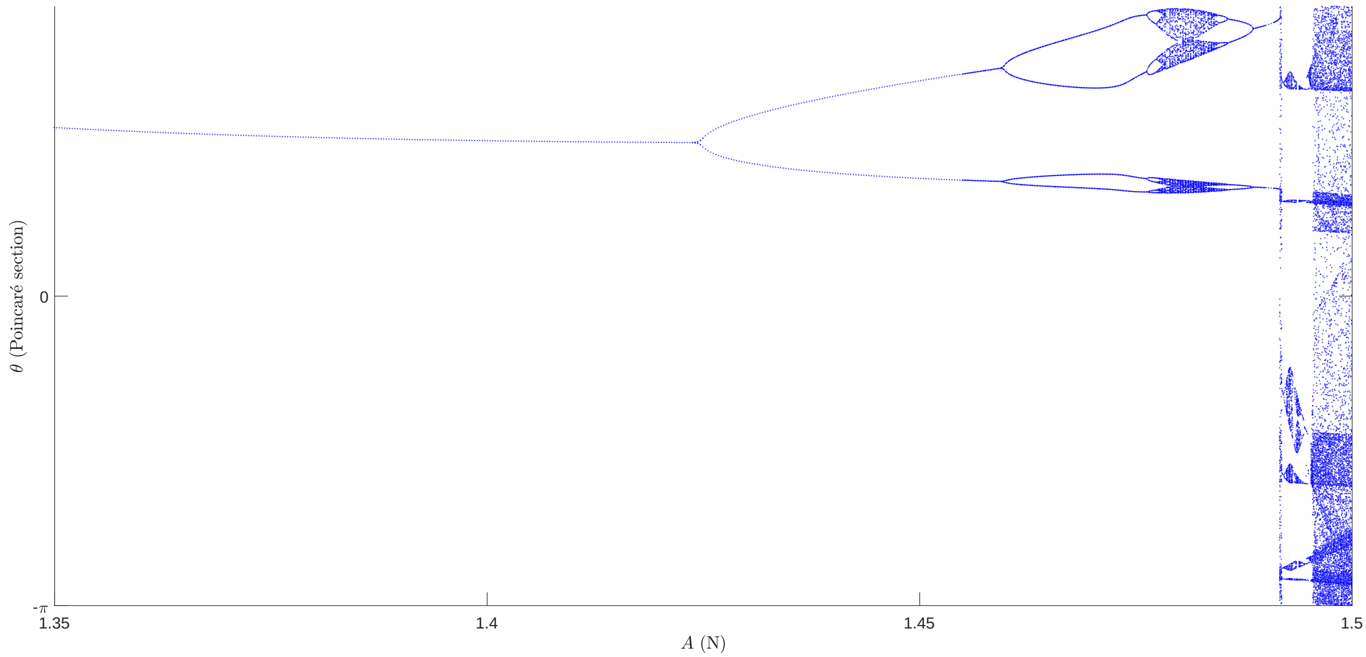


Figure 6.1: The transistion to chaos for $1.45$ N$<A<1.5$ N. The graph shows the bifurcation diagram of the long time Poincaré section of $\theta$ vs $A$ . Enlargements of regions of this graph are given in figures 6.2 and 6.3.

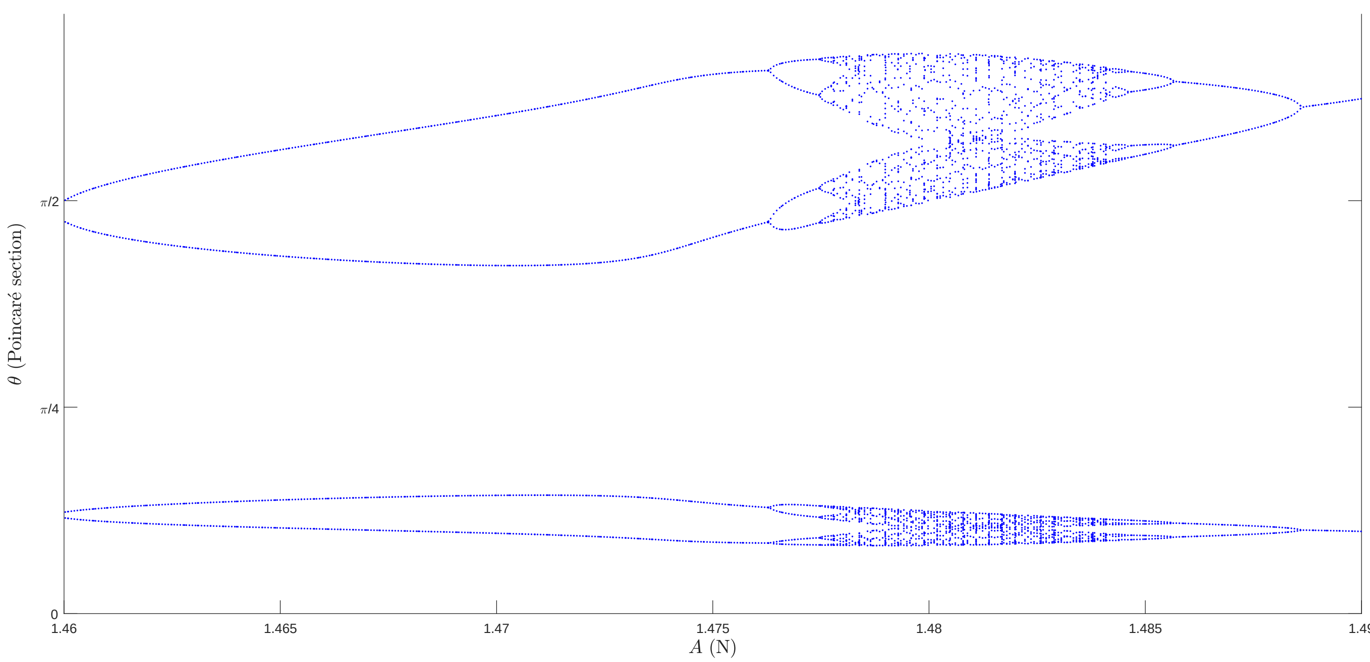


Figure 6.2: An enlargment of part of figure 6.1.

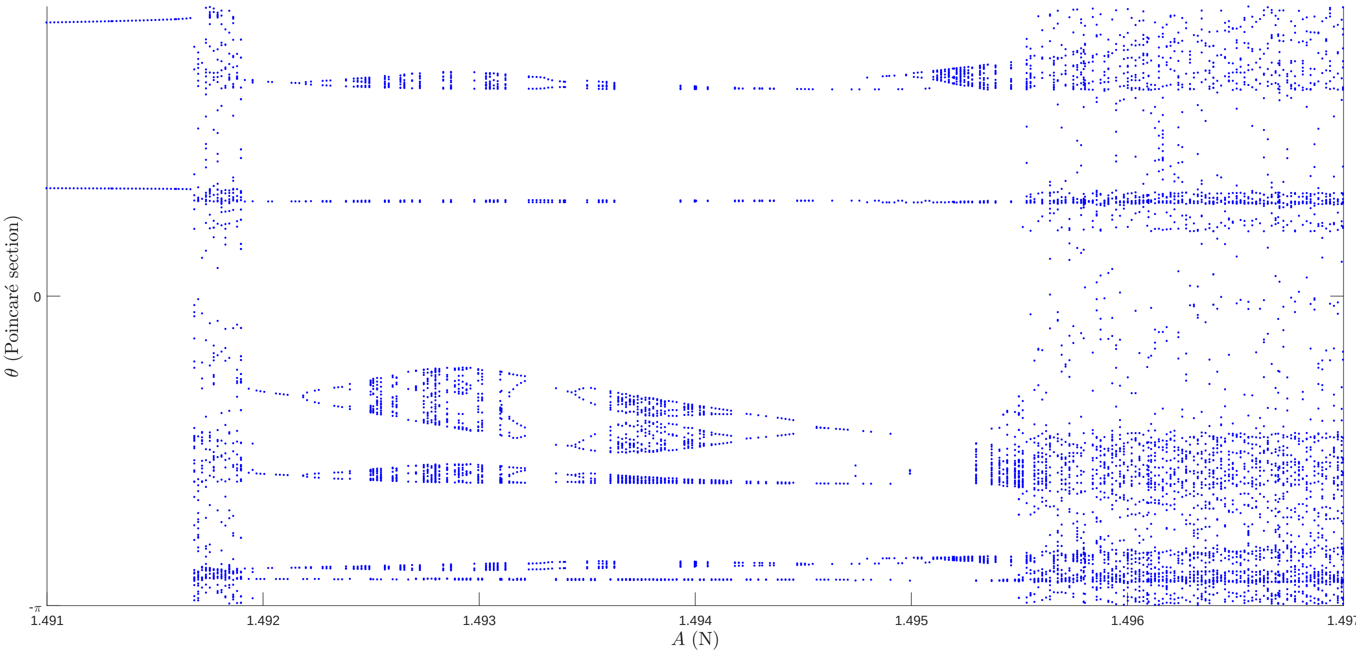


Figure 6.3: An enlargment of part of figure 6.1.

### $0.5$ **N**$< A < 1.2$ **N**

The script `Part6.m` relates to this subsection.

The code calculates 50 Poincaré points $\theta(Tn)$ are after the pendulum has be in motion for 300 driving periods for different values of the driving amplitude. A plot of the Poincaré points vs the driving amplitude is shown in the main graph of figure 6.1. Looking at the graph we see that there is one periodic orbit whose Poincaré section changes smoothly till about $A \approx 1.04$ N where it "jumps" suddenly and it jumps back down at about $A \approx 1.06$. We can understand what has happened here by looking at the full phase space plot orbits either side of the jump shown bottom left graph we see that the jump between two different stable periodic orbits with the same period, it turns out the stable orbit has bifurcated into two stable orbits at about $A \approx 1$ N (we could see both if we combined Poincare sections with multiple different initial conditions). A further series of these bifurcations some accompanied by period doubling appear to occur before $A = 1.102$ N where there is an interesting looking stable orbit with a period three times that of the driving (bottom right graph in red) then chaos. This is followed by another window of order where there are periodic orbits then more chaos.

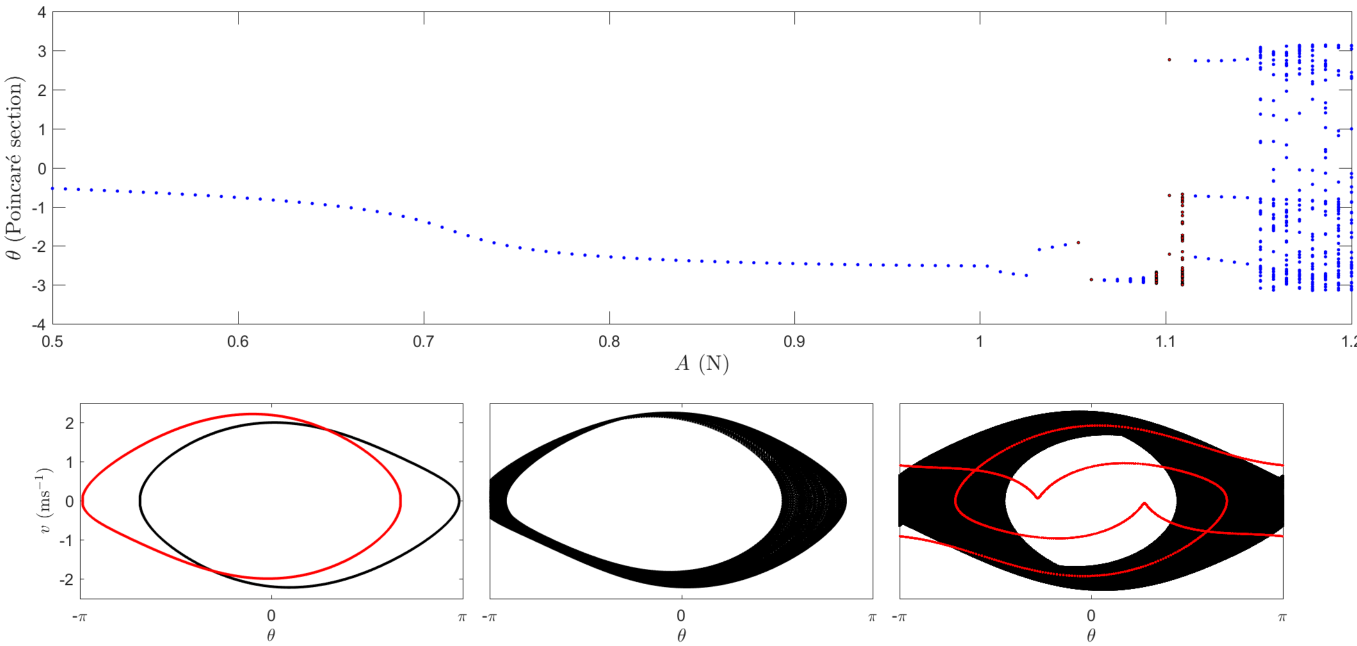


Figure 6.4: The transition to chaos for $0.2$ N$<$ $A$ $<$ $1.2$ N. The top graph shows the bifurcation diagram of the long time Poincaré section of $\theta$ vs $A$ (blue and red symbols). The red symbols have long time phase portraits shown below. In the bottom left plot we have the phase portraits with $A = 1.053$ N in black and $A = 1.06$ N in red. In the bottom centre plot we have phase portrait with $A = 1.095$ N. In the bottom right plot we have the phase portraits with $A = 1.102$ N in red and $A = 1.109$ N in black.