410 Tutorial 8: Differential Equations

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The goal of this tutorial is to become familiar with Runge-Kutta methods for solving ordinary differential equations numerically.

1 Runge-Kutta Methods

Suppose we want to solve a differential equation of the form,

$$\frac{dy(t)}{dt} = f(y(t), t) \tag{1}$$

The most naive way to find the solution to (1) numerically is to divide the integration region into steps of size h and find the value of $y_{i+1} = y(t_{i+1})$ by using the information at step i:

$$y_{i+1} = y(t_i + h) \approx y(t_i) + hf(t_i, y_i) \tag{2}$$

This method is known as the simple Euler method. It works in practice, but not miraculously. Indeed, the simple Euler method assumes that the derivative at the beginning of the interval remains constant over the entire step, which results in poor accuracy. The insight of Runge and Kutta was to improve the approximation to the derivative by sampling the function $f(t, y)$ at many intermediate steps between (t_i, y_i) and (t_{i+1}, y_{i+1}). Using such intermediate quantities effectively corrects the overshoot we make when using other simpler methods. The fourth-order explicit Runge-Kutta method, often called RK4, uses 4 intermediate quantities to reach fourth-order accuracy[1]. Given information at some time (t_i, y_i), we find the solution at the next time step t_{i+1} to be,

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{3}$$

[1] The local truncation error of RK4 goes like $O(h^5)$ whereas the total error after integrating for $\approx \frac{1}{h}\, steps$ is $O(h^4)$

where,

$$k1 = f(t_i, y_i) \tag{4}$$
$$k2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_1) \tag{5}$$
$$k3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}k_2) \tag{6}$$
$$k4 = f(t_i + h, y_i + hk_3) \tag{7}$$

The procedure is then repeated at every time step until we have solved the differential equation on the desired interval $[t_0, t_f]$.

2 Simple Harmonic Oscillator

The simple harmonic oscillator is a good starting point for numeric integrations. The analytic solution is well known and simple, so it is easy for verify your code and check to see that we are getting the desired rates of convergence (more on this in the next tutorial and in class).

$$\frac{d^2y}{dt^2} = -y \tag{8}$$

1. Write 8 as a system of two first-order differential equations.
2. Use RK1, RK2 and RK4 to integrate the solution corresponding to $y(0) = 1$, $\frac{dy(0)}{dt} = 0$ on the interval $t = [0, 10\pi]$.
3. How does each RK method compare with the analytic result?

3 Lorenz System

The Lorenz system is a system of ordinary differential equations first studied by Edward Lorenz. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system which, when plotted, resemble a butterfly or figure eight. As a differential equation, the Lorenz system is given by,

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned} \tag{9}$$

where σ, ρ and β are system parameters.

1. Write the system 9 as a Matlab routine.
2. Choose $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$
3. Plot the behaviour of the system for a variety of initial conditions
4. Choose two initial conditions that differ only by some small number and plot the magnitude of the divergence between those paths. Describe the rate of divergence.