410 Tutorial 11: Partial Differential Equations: Stability

November 2017

1 Courant–Friedrichs–Lewy condition

The Courant–Friedrichs–Lewy (CFL) condition is a necessary condition for convergence while solving certain partial differential equations (typically hyperbolic PDEs). It arises in the numerical analysis of explicit time integration schemes, when these are used for the numerical solution. As a consequence, the time step must be less than a certain time in many explicit simulations, otherwise the simulation produces incorrect results.

Consider a hyperbolic differential equation which admits a mode which travels with velocity u on a grid spaced by Δt in time and Δx_i in each of n spatial dimensions. Let the velocity (relative to the grid) of the fastest mode be c, our "speed of light" or "speed of causation." Then, we expect that if,

$$\sum_{i}^{n} \Delta x_i < c\Delta t \tag{1}$$

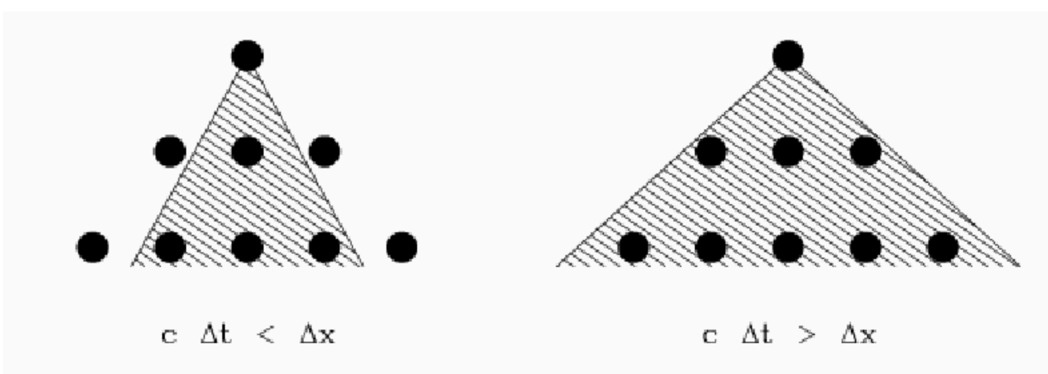

Figure 1: Illustration of the CFL condition

then, when we apply our spatial stencil for evolution involving nearest neighbour points, our solution requires information from outside the spatial interval as shown in Figure 1 for the 1D case. Therefore, in general for a hyperbolic equation, we require,

$$\Delta t \sum_{i}^{n} \frac{u_i}{\Delta x_i} = C \leq C_{max} \tag{2}$$

Where C_{max} changes with the numeric integration method and spatial stencils used (Consider how the situation changes if we use a spatial finite difference stencil of various widths).

2 Von Neumann stability analysis

Unfortunately, as shown in the previous tutorial with the viscus Burger's equation, simply obeying the CFL condition is insufficient to guaranty stability. One can get a much better idea of numeric stability through the use of Von Neuman stability analysis. The procedure is simple and is illustrated here in detail for the case of the advection equation evolved using Euler's method.

$$u_t = cu_x \tag{3}$$

$$u_i^{n+1} = u_i^n + \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) \tag{4}$$

Assuming that we have a numeric solution $N(x,t)$ related to the true solution $u(x,t)$ by,

$$u_i^n = N_i^n - \epsilon_i^n \tag{5}$$

where epsilon is our error term. Since $N(x,t)$ satisfies our equation by assumption, we have,

$$\epsilon_i^{n+1} = \epsilon_i^n + \frac{c\Delta t}{2\Delta x}\left(\epsilon_{i+1}^n - \epsilon_{i-1}^n\right) \tag{6}$$

now, without loss of generality, we can expand $\epsilon(x,t)$ in Fourier basis functions. At the same time, lets assume that the error terms either oscillate or grow exponentially in time:

$$\epsilon(x,t) = \sum_{m=1}^{\frac{L}{\Delta x}} e^{a_m t} e^{ik_m x} A_m \tag{7}$$

$$k_m = \frac{\pi m}{L} \tag{8}$$

Since our equation for $\epsilon(x,t)$ is linear, we can look at each mode separately,

$$\epsilon_i^n = e^{a_m t} e^{ik_m x} \tag{9}$$
$$\epsilon_i^{n+1} = e^{a_m t+\Delta t} e^{ik_m x} \tag{10}$$
$$\epsilon_{i+1}^n = e^{a_m t} e^{ik_m (x+\Delta x)} \tag{11}$$
$$\epsilon_{i-1}^n = e^{a_m t} e^{ik_m (x-\Delta x)} \tag{12}$$

plugging these in and simplifying,

$$e^{a_m t} = 1 + \frac{c\Delta t}{2\Delta x}\left(e^{ik_m \Delta x} - e^{-ik_m \Delta x}\right) \tag{13}$$
$$e^{a_m t} = 1 + \frac{ic\Delta t}{\Delta x}\sin(k_m \Delta x) \tag{14}$$

Now, for stability, we require that the real part of a is less than or equal to 0 such that the error is decreasing in time.

$$\left|e^{a_m t}\right|^2 = e^{a_m t} e^{a_m^* t} <= 1 \tag{15}$$
$$0 \leq \frac{c\Delta t}{\Delta x}\left(1 - \cos^2(k_m x)\right) \tag{16}$$

Evidently, the advection equation with Euler time integration is always unstable. Note that this analysis likewise indicates that backwards Euler integration is always stable.

3 Final Activity: Swift–Hohenberg equation

The swift-Hohenberg equation appears in the context of thermal convection and is well known for its pattern forming behaviour.

$$u_t = \alpha u - \left(1 + \nabla^2\right)^2 u + u^2 - \beta u^3 \tag{17}$$

1. Perform a Von Neumann stability analysis for the 1D Swift–Hohenberg equation with an Euler integrator. Is it stable? Does adding a dissipative term help?

2. Write a solver for the 2D Swift–Hohenberg equation. If you implement it using a explicit integrator, what conditions do you expect to be placed on the time step?

3. If you have time, work this out explicitly using a Von Neumann stability analysis for Crank Nicholson time integration.

4. Investigate the behaviour of the system with random initial data and periodic boundary conditions for the following values of the parameters. What sort of paterns do you observe?

5. $\alpha = 0.3, \beta = 0$

6. $\alpha = 0.1, \beta = 1$