

# Fourier Analysis

## Introduction:

This experiment is intended to give an introduction to the conventional technique used in the analysis of waves. The concept is based on the Fourier Theorem, which states that all waves, no matter how complex can be expressed as a sum of sines and cosines. Essentially, this means we can break up a complex waveform into simpler parts, and all we need do is to add these simple parts together to obtain the original wave back. Once the complex wave is broken up, we can analyze the results using the simpler sine waves and obtain the final answer in a greatly simplified manner.

## The Fourier Transform

In this experiment we will be working with functions of time. Fourier analysis can be done on functions of anything, e.g. distance, magnetic field, light intensity, etc, but the procedure is identical to that which we describe in this lab. The Fourier transform is a plot of amplitude vs. frequency. It tells us how much sine wave at a particular frequency is present in our time function. If we start with a function of time  $g(t)$  the Fourier transform is  $G(f)$ . A few examples should make this clear. Figure 1 a shows a simple sinusoid  $g(t)$  and its Fourier amplitude spectrum  $G(f)$ .  $G(f)$  has a spike at the frequency of the sinusoid in  $g(t)$ . If  $g(t) = \sin 2\omega_1 t$ , the spike would have the same amplitude, but would be shifted to  $2f_1$  on the  $f$ -axis. If  $g(t) = 3 \sin 2\omega_1 t$ , the spike would still be at  $2f_1$  on the axis, but the amplitude of the spike would become 3. Figures 1b and 1c should be self-explanatory

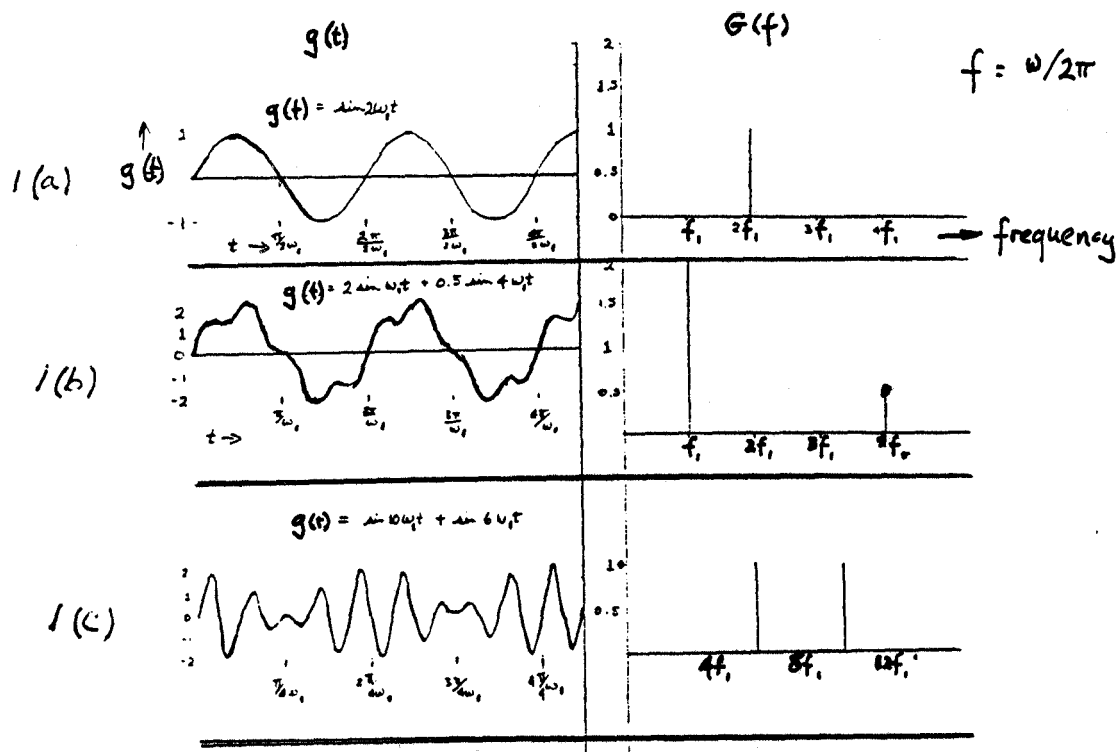


Figure 1.

Consider a square wave as shown in Fig. 2.

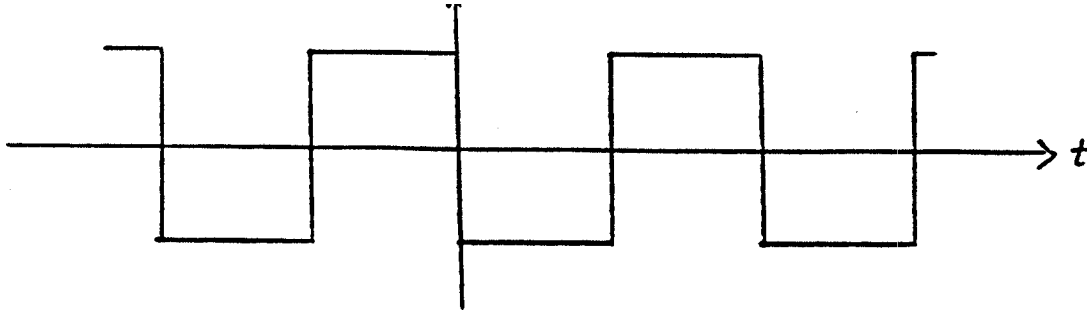


Figure 2.

One can show that  $g(t) = 4/\pi [ \sin \omega_1 t + 1/3 \sin 3\omega_1 t + 1/5 \sin 5\omega_1 t + \dots ]$  (Eq. 5.1). This is shown in Fig. 3, where some of these terms are plotted.

$$S(1) = \sin \omega_1 t$$

$$S(2) = \sin \omega_1 t + 1/3 \sin 3\omega_1 t$$

$$S(3) = \sin \omega_1 t + 1/3 \sin 3\omega_1 t + 1/5 \sin 5\omega_1 t,$$

etc...

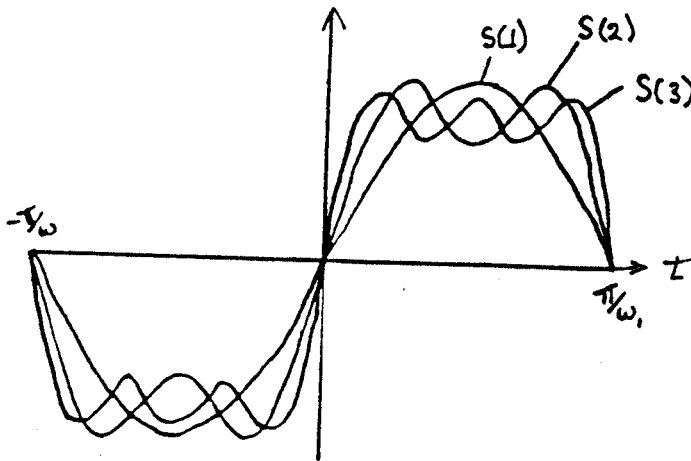


Figure 3.

One very common application of the Fourier transform is the determination of the response of a system to a time varying stimulus  $g_i(t)$ .  $g_i(t)$  can be very complicated, and it would be difficult to analyze the response of the system to it directly. Sine waves are easy to analyze, and if we break up  $g_i(t)$  into the sine waves which constitute it, we are left with the much simpler sine wave analysis. This breaking up of a stimulus,  $g_i(t)$ , into an equivalent sum of sine waves is the major undertaking of this experiment.

## The Calculation of the Response

In order to use the Fourier transform to help us in this calculation we need the input/output relationship that expresses the “linearity” of the system:

$$G_o(f) = G_i(f) H(f) \quad (\text{Eq. 5.2})$$

Where  $G_o(f)$  and  $G_i(f)$  are the Fourier transforms of the output response and the input stimulus respectively.  $H(f)$  is called the “Transfer Function” and can be determined in many ways. Note that the Transfer Function comes out as a Fourier transform, so that if we want the time dependence, we would have to go backwards and take an inverse Fourier transform. Most of the time, we can determine everything we need from  $G_o(f)$  and  $g_o(t)$  is not needed.

The Transfer Function can be determined by looking at the response of the system to a white noise source. The Fourier transform of this response is  $H(f)$ , or the system transfer function. (White noise is that which has a flat Fourier amplitude spectrum. This means that all frequencies are present in the noise with equal amplitude. The result is a noise which looks very random and never repeats.)

Another way to determine  $H(f)$  is to look at the response of the system to a short, very strong impulse. The Fourier transform of this response is also  $H(f)$ .

The laboratory is equipped with a digital oscilloscope that can act as a spectrum analyzer, as well as with a computer system, either of which can take the Fourier transform of an electrical signal. With these instruments, the job becomes much easier, since it eliminates the most tedious part i.e. the actual calculation of the Fourier transform.

To illustrate the method of calculating the response, let us take an example. Consider a suspension bridge. A suitable stimulus is a person on the bridge, running on the spot. Let us consider one point on the bridge (P) and calculate its vertical displacement after the runner begins.

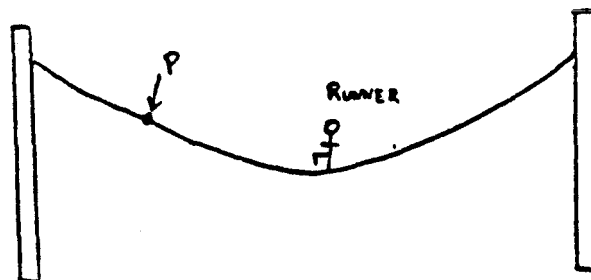


Figure 6. Runner on a suspension bridge.

The first step is to find the Fourier spectrum of the stimulus. So, have the runner run on a strain gauge (force measuring device) and have the analyzer find the Fourier spectrum. Now, we need not consider Fig. 7a, the force as a function of time, but it is included for completeness.

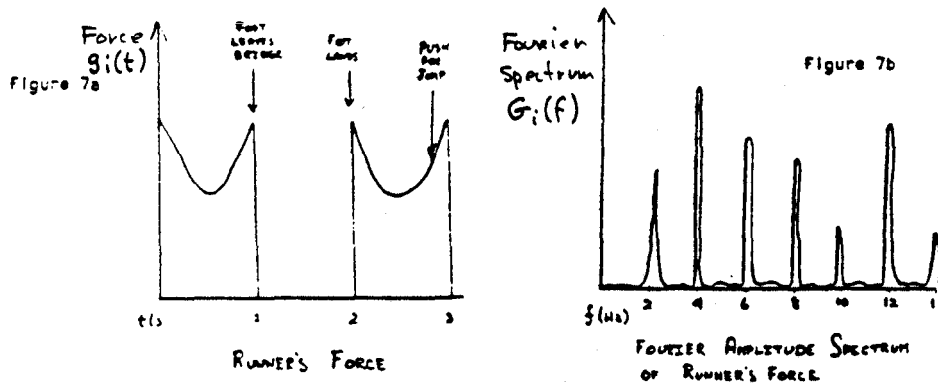


Figure 7. (a) Runner's force on bridge (b) Fourier amplitude spectrum of runner's force

The second step is to get the bridge's response, so, we attach a displacement-measuring device at point P, and plug it into our Fourier transform analyzer. Then, we can either hit the bridge with a hammer and measure the response, or we can attach a speaker and electronically feed it white noise, and find the response. In either case, the response is the same, and the Fourier spectrum will resemble Fig. 8, it is the Transfer function.

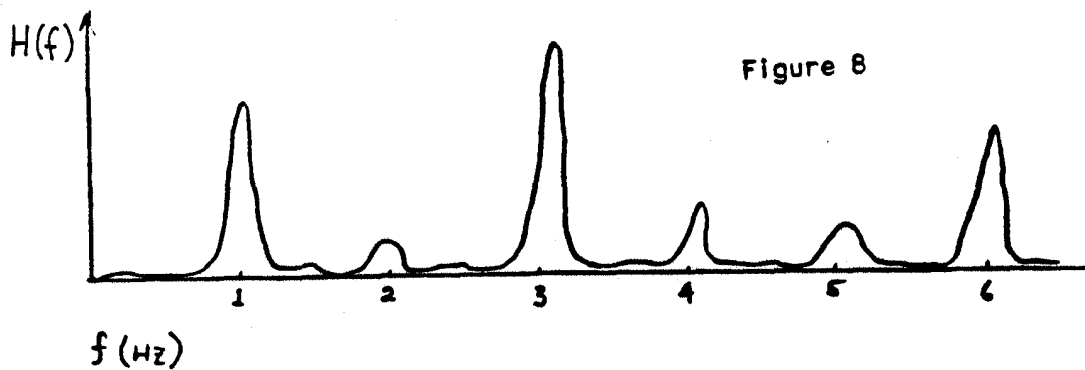


Figure 8. Transfer function of bridge.

Our result upon the multiplication of Fig 7 (b) (the stimulus  $G_i(f)$ ) and Fig 8 (The transfer function of the bridge  $H(f)$ ) is shown in Fig. 9.

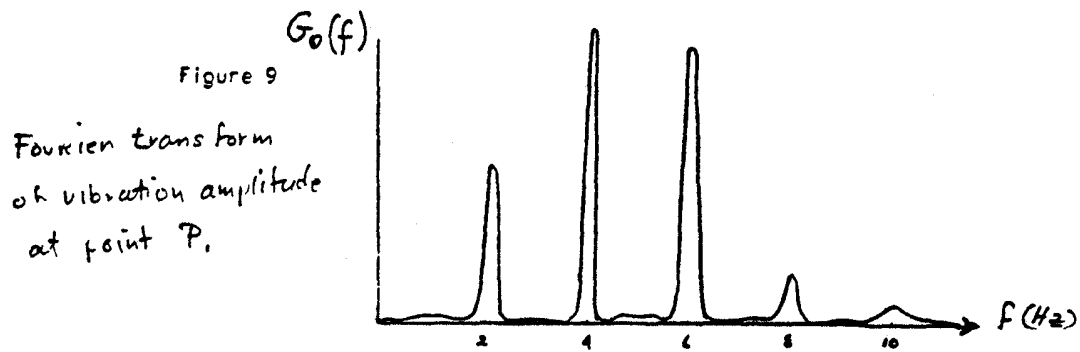


Figure 9. Response of bridge to force of runner.

So we can say that the bridge will vibrate mostly at 4 Hz and 6 Hz. Actually, it is possible to figure out the exact time dependence of the motion, but this transform is usually all one needs.

We can now tell the runner to run at a frequency which makes the bridge move with a large amplitude simply by matching up one of the peaks in the runner's Fourier Transform with one of those in the bridge's response. From Fig. 8, we can tell the runner to jog at 1 Hz, which lines up the first peaks in Fig. 7b and Fig. 8. When we multiply them we get a large answer. This is simply one of the "Resonant Frequencies" of the bridge. If the runner runs at 1/2 Hz, the second peak in Fig. 7b lines up with the first in Fig. 8 resulting in even a larger response. Everyone is familiar with rocking a bridge. If you stand on the bridge and rock at just the right frequency, the bridge vibrates with large amplitude. The frequency at which the spike appears is the frequency of the force. When you rock, you produce very close to sinusoidal force. Its Fourier spectrum is a single spike.

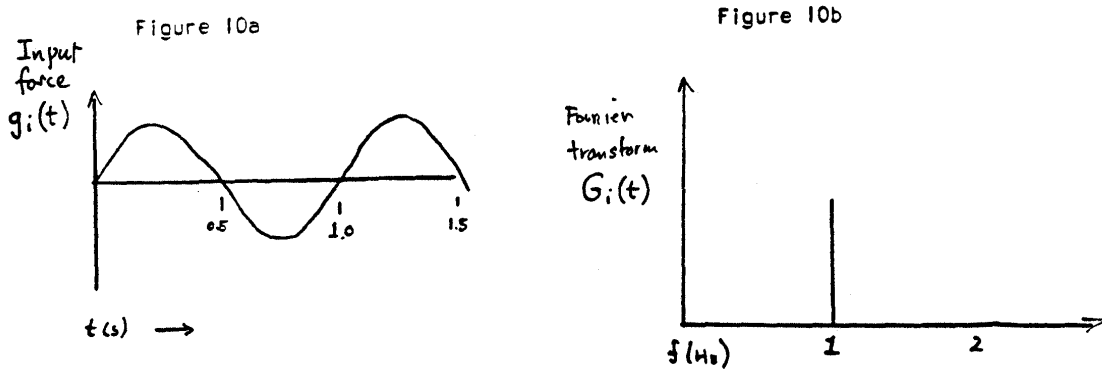


Figure 10.

If Figure 8 is multiplied by Figure 10 (frequency by frequency) in accordance with equation 5.2 the result is very dependent on the driving frequency  $f_i$ . If  $f$  lines up with a large peak of Figure 8 then the bridge goes wild; if not, then it still moves at frequency  $f_i$  but with a smaller, possibly negligible, amplitude.

## Apparatus

The system consists of two identical loudspeakers facing each other and linked by direct acoustical transmission and by a light coiled spring with natural frequencies in the 1 Hz range. The loudspeakers each have a natural oscillation frequency below 100Hz.

Signals may be applied to the right-hand end via the two-pin connector. The signal source may be a single pulse source, the sound-card output of the computer, or an amplified version of this sound-card output (amplified by the powered computer speakers).

The response of the system  $g_o(t)$  is observed at the opposite end to the drive. The response signal arises from the fact that a loudspeaker is a reversible transducer - when the loudspeaker is driven mechanically, the forced oscillation of the coil in the magnetic field of the magnet gap causes a corresponding electrical signal through electromagnetic induction.

There are several ways of doing the Fourier transform. The HP oscilloscope can do an FT internally, or we can use the HP scope to log data which can be read into the computer, and do the FT on the computer. We'll use each strategy in different parts of the experiment.

## Prelab Questions

Read through the lab and pay special attention to the introductory parts about Fourier series and how to write periodic time-functions as sums of sines (for odd functions) and cosines (for even functions). We'll use odd functions in this assignment. The extension to a general periodic time function is trivial.

**QUESTION** An odd, periodic function can be written in the form:

$$g(t) = \sum a_n \sin(\omega_n t), \quad n = 1, \infty.$$

Where  $\omega_n = 2\pi n f$  and  $T (=1/f)$  is the period of the function  $g(t)$ .

The coefficients  $a_n$  can be obtained as follows:

$$a_n = 2/T * \int_0^T (g(t) \sin(\omega_n t)) dt$$

For a function that requires several sines (i.e., several terms  $a_1 \sin(\omega_1 t) + a_2 \sin(2\omega_1 t) + \dots$ ), you simply fix  $n$  and find the coefficients in turn. This way,

$$\begin{aligned} a_1 &= 2/T * \int_0^T (g(t) \sin(2\pi t/T)) dt \\ a_2 &= 2/T * \int_0^T (g(t) \sin(4\pi t/T)) dt \quad \text{etc.} \end{aligned}$$

- Show that the coefficient  $a_1 = 1$  for  $g(t) = \sin(2\pi t/T)$ .  $T$  can be arbitrary, but you can use a finite value if you like.
- Calculate the first three non-zero coefficients for the square-wave stimulus function:  $g(t) = -1/2, t \in [-T/2, 0]$ ;  $g(t) = 1/2, t \in [0, T/2]$ . To do this, simply split up the integral from  $t=[-T/2, 0]$  and  $t=[0, T/2]$ . Plot the sum of these first three terms. My definition of  $g(t)$  here is simply to avoid a DC-term, and to ensure that the function is odd.
- Repeat part (b) for a triangular function. You need to define this function; just make sure that it is odd. Get the first three non-zero terms. Plot the sum of the three first terms.
- Calculate the error in the case of the stimulus being the input (part (b)). Do this by calculating the integral of the difference of  $g(t)$  and the approximation squared over a full period:  $\text{error} = \sqrt{1/T * \int_0^T ((g(t) - \text{approx})^2) dt}$ . Calculate the error for only one term, then two (non-zero) terms, then three. Comment on your findings.

Some instructions on how to do this assignment with Maple are given below. Note that if you for some reason cannot get Maple to work, then do this assignment by hand or with Mathematica. The integrals involved are fairly straightforward. Part (c) might require integration by parts.

### How to do integrals with Maple.

This term, we'll introduce Maple. This is a mathematical software package that lets you do symbolic and numerical mathematics.

Start Maple on a physics UNIX terminal by typing at the prompt:

→ **xmaple &**

This will open a GUI (graphical user interface). Define  $g(t)$  in question (a) by:

→ **g:=sin(2\*Pi\*t/T);**

The semicolon at the end allows you to see the output. A colon will remove it. Note the colon immediately before the equal sign. This is mandatory in a function definition. Note the capital "P" in "Pi". Now, we have to tell Maple that T is positive:

→ **assume(T>0);**

Now, do the integral:

→ **int(2/T\*g\*sin(2\*Pi\*t/T), t=0..T);**

The answer will pop up in a second. You might have to reorganize the result a little. (In this case, the answer should be 1, right?).

A plot is easily generated in Maple. I.e., to see our function  $g$ , we type:

→ **g2:=subs(T=2,g);**

This simply substitutes the value 2 in for T in  $g$  and redefines it as the function  $g2$ .

→ **plot(g2, t=-3..3);**

If you require help in Maple, for example on integration, then type "help(int);". To reset all variable, etc., type "restart;" on the command line.

### Things to look up.

A few websites that let you play with these concepts can be found at:

<http://www.phy.ntnu.edu.tw/java/sound/sound.html>,

<http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCEXamples/Fourier/fourier.html>, and <http://www.jhu.edu/~signals/listen/music1.html>.

The last one is most excellent, but requires a soundcard for full effect. The files for download are also on the order of ~1MB, so I hope you have a fast Internet connection. The second site is a UBC site, and has lots of animations of different mathematical phenomena. You can find other sites by searching for "fourier series" in your favourite search engine.

## **Procedure**

### **Part I** - Familiarization with the Spectrum Analyzer and the Fourier Transform.

1. Provide a sinusoidal input (from the function generator) to the HP oscilloscope and use the scope to take the FT. See the technical notes page for some details on setting the scope up. Note the effect on the FT to changes of amplitude of the signal input.
2. Feed a square-wave signal to the analyzer. Measure the frequencies and amplitudes of the spikes of the FT and so check that the FT conforms to equation (5.1). Observe what happens to the FT when the frequency of the signal is changed.
3. Feed a triangular waveform to the analyzer. Compare and contrast the FTs for sine wave, square-wave and triangular-wave at the same fundamental frequency.
4. Repeat some of the above measurements with the function generator still connected directly to the oscilloscope but now using the computer to retrieve the data from the scope, doing the FT on the computer. See the technical notes page for details on how to do this. Collect at least one data set for a sine wave input, a square wave input and a triangle wave input.

### **Part II** - The amplitude response of a complex system.

1. Establish the Transfer Function  $H(f)$  for the loudspeaker system –
  - a. Apply white noise and take the FT of the system response for the frequency range 0-100 Hz. Plot the result. Observe the FT as a function of noise amplitude. Non-linear effects can appear at large amplitude; do you observe any such effect? For this part, we use the computer sound card as the signal source – see the technical notes.
  - b. Apply an impulse spike to the system, from the supply provided. The pulse is generated by the discharge of a capacitor.. The pulse width should be small compared with the period of the highest frequencies of interest. (Why?) Plot the FT of the system response and compare the result with the white noise response. For this part, you'll want to ensure that the oscilloscope triggers on the spike and then doesn't trigger again till the next one. To improve the signal to noise ratio of this measurement, you can use the signal averaging function of the oscilloscope (found by pressing the display button) and repeat the measurement several times.
2. From the Transfer Function plots, find the frequency at which the system resonates with the greatest amplitude. Find the frequencies of two or three other peaks, and estimate their heights relative to the first peak. Now apply sine-wave signal inputs at these frequencies and record the response amplitudes. Make sure that you really are observing the peaks - search over a small frequency range to find the maximum response. Do the observed relative amplitudes agree with the ratios expected from the Transfer Function?
3. Apply a square-wave signal to the system for a fundamental frequency for which you expect a significant response; plot the FT of the response. Compare the result with what you would



expect on the basis of equation (5.2) from the known Transfer Function and the known spectrum of the input signal as given by (5.1). Repeat for a different signal frequency.

Getting your data off the computer will require either a USB flash drive.

**In general you should always archive your raw data – not just the analyzed versions of it, in case you need to revisit your analysis later on.**