Basic Error Analysis for Experimental Physicists

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1 Introduction

This handout will cover the basics of estimating errors associated with measurements, as well as calculations of propagating errors.

For further reading, please search the internet or check out the following references:

1. H. Young, Statistical Treatment of Experimental Data, McGraw-Hill, New York, 1952.

2. J. Taylor, An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements, University Science Books, Mill Valley, CA, 1982.

3. N. Bardord, Experimental Measurements: Precision, Error and Truth, Addison-Wesley, Reading, MA, 1967.

4. D. W. Preston and E. R. Dietz, The Art of Experimental Physics, John Wiley & Sons, Inc., 1991.

2 Statistical Analysis of Random Errors

If a physical quantity, such as a length interval measured with a meter stick or a time interval measured with a stopclock, is measured **many times**, then a **distribution** of readings is obtained because of random errors. For such a set of data, the average or mean value $\bar{x}$ is defined by:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1}$$

where x_i is the ith measured value and n is the total number of measurements. If there are no systematic errors, then $\bar{x}$ will approach the "true value" as $n \to \infty$ (that means "large").

We estimate the error or uncertainty in this value. The **standard deviation** σ is defined as:

$$\sigma \equiv \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \tag{2}$$

where σ is in the same units as x_i. The error or uncertainty in the mean value $\bar{x}$ is the **standard error**, σ_m, which is defined as:

$$\sigma_m \equiv \frac{\sigma}{n^{1/2}} \tag{3}$$

and finally, you quote your measurement as:

$$\bar{x} \pm \sigma_m \tag{4}$$

This indicates that the measured values lie in the range from $\bar{x} - \sigma_m$ to $\bar{x} + \sigma_m$ with a $\sim 67\%$ probability. We will refer to σ_m as the "absolute error", or the "uncertainty" from here on.

3 Estimating Experimental Errors: Should I use the Manufacturer's Estimate?

When you know the manufacturer's estimate of what a measuring device's uncertainty is, that's often the best value to use. However, in some cases, even these uncertainties might be too optimistic. For example, a manufacturer of stopwatches might list the accuracy of its devices as $\pm$ 0.005 seconds. However, the average human being has a reaction time of about 0.2 seconds. That means that the appropriate uncertainty for a stopwatch triggered by hand is $\pm \sim$ 0.28 seconds ($\pm$ 0.2s for starting it at the correct time, and $\pm$ 0.2s for stopping it at the correct time, added in quadrature; see later). The manufacturer's listed uncertainty is too optimistic for measurements taken by hand.

Unfortunately, there are no fixed rules for determining what the uncertainty on a measured value should be. You must decide what uncertainty value you think will best reflect the true accuracy of your measurement. The question you have to ask yourself is: "How accurately can I read off the value based on this setup?", i.e. for a meterstick: the division will limit your accuracy ($\pm$ half the smallest division is OK), for an oscilloscope: the line is blurry, and half the smallest division is 0.2div; perhaps $\pm$0.1div is good? When there's moving quantities involved, then the error will most likely be larger (like measuring the time at which a pendulum hits a certain point in its cycle).

One can record errors as either:

Absolute errors: These are errors WITH units, like 0.3m in 6.0±0.3m.

Relative or percentage errors: These are errors WITHOUT units, like 0.05 (equivalent to 5%) in the expression above: 0.3nm/6.0nm = 0.05 → 5%.

4 Errors propagating through functional relationship

The equation that governs it all is the following (assuming that all the variables are uncorrelated):

$$df^2 = \left(\frac{\partial f}{\partial x}\right)^2 dx^2 + \left(\frac{\partial f}{\partial y}\right)^2 dy^2 + \left(\frac{\partial f}{\partial z}\right)^2 dz^2 + \cdots \tag{5}$$

where $f = f(x, y, z, ...)$ is the function for which we want to find the absolute error df and $(dx, dy, dz...)$ are the estimated errors associated with each variable $(x, y, z...)$.

4.1 An Example of Propagating Errors.

Let $f = xln(y^2/z) + \sqrt{x^3 z}$, so $f = f(x, y, z)$. We assume that the values for (x, y, z) are known, along with the absolute errors (dx, dy, dz). We get:

$$\begin{aligned} df^2 &= \left(\frac{\partial f}{\partial x}\right)^2 dx^2 + \left(\frac{\partial f}{\partial y}\right)^2 dy^2 + \left(\frac{\partial f}{\partial z}\right)^2 dz^2 \\ &= \left(\log\left(y^2/z\right) + 3\sqrt{xz}/2\right)^2 dx^2 + (2x/y)^2\, dy^2 \\ &\quad + \left(-x/z + \sqrt{x^3/4z}\right)^2 dz^2 \end{aligned} \tag{6}$$

We complete this with a numerical extension: Let:

$$\begin{aligned} x &= 1.5 \pm 0.1m^2, \\ y &= 18.0 \pm 0.6m^{-1}, \\ z &= 3.20 \pm 0.03m^{-2}, \end{aligned} \tag{7}$$

We get:

$$df^2 = \left(\log\left[(18m^{-1})^2/3.2m^{-2}\right] + 3\sqrt{1.5m^2 \times 3.2m^{-2}}/2\right)^2 (0.1m^2)^2 \tag{8}$$

$$
\begin{aligned}
& + \left(21.5m^2/18m^{-1}\right)^2 (0.6m^{-1})^2 \\
& + \left(-1.5m^2/3.2m^{-2} + \sqrt{(1.5m^2)^3/4 \times 3.2m^{-2}}\right)^2 (0.03m^{-2})^2 \qquad (9) \\
= & \ (0.625 + 0.010 + 1.801 \times 10^{-6})m^4 \simeq 0.635m^4 \qquad (10)
\end{aligned}
$$

Note that the largest contributor to the total error in f is the uncertainty due to x, whereas z contributes the least to the overall uncertainty. (We could have guessed that from looking at the relative error of the three variables dx/x, dy/y and dz/z). Then, $df = \sqrt{0.635m^4} = 0.797m^2$, so

$$f = 10.213 \pm 0.797m^2 \simeq \underline{10.2 \pm 0.8m^2} \qquad (11)$$

Here, we leave one significant figures in the error, and match the last sig. fig. in f to the error (more about sig. fig.'s later).

5 More Specific Examples and Formulas

For an equation $f = f(x, y, z...)$, with a and b being constants, we have the following specific formulae:

$$
\begin{aligned}
f = ax \pm by &: \ df^2 = (adx)^2 + (bdy)^2 \qquad (12) \\
f = \pm axy &: \ \left(\frac{df}{f}\right)^2 = \left(\frac{dx}{x}\right)^2 + \left(\frac{dy}{y}\right)^2 \qquad (13) \\
f = \pm \frac{ax^n}{y^m} &: \ \left(\frac{df}{f}\right)^2 = \left(n\frac{dx}{x}\right)^2 + \left(m\frac{dy}{y}\right)^2 \qquad (14) \\
f = ax^{\pm b} &: \ \frac{df}{f} = b\frac{dx}{x} \qquad (15) \\
f = ae^{\pm bx} &: \ \frac{df}{f} = bdx \qquad (16) \\
f = aln(\pm bx) &: \ df = a\frac{dx}{x} \qquad (17)
\end{aligned}
$$

6 How to Minimize Errors Through Your Measurement

Your estimated error depends on how you do your measurement. For example, when reading off a voltage from the 'scope. Say that each voltage reading has a $\pm 1/2$(the smallest scale division) uncertainty. The smallest scale division in the case of the 'scope is

0.2div. Then, by making a measurement of the amplitude $V_A = 3.2$div, we have an error $dV_A = \sqrt{(0.1div)^2 + (0.1div)^2} = 0.1414$div. Consider, however, that we measure the peak-to-peak voltage, instead. We would then get the same absolute error of $dV_{pp} = 0.1414$div, however, we can extract the amplitude $V_A = V_{pp}/2$ and then the error $dV_A/V_A = dV_{pp}/V_{pp}$, or since $V_A = V_{pp}/2$, we get $dV_A = dV_{pp}/2$, so the error in our amplitude measurement has been halved from the initial measurement: $dV_A = 0.0707$div. Similarily, if you're want to measure the period of a periodic signal on the 'scope: measure for example three periods, and divide by 3. This will reduce your reading error.

7 Significant Figures

The overriding principle that applies to rounding decimal numbers is this: you never want to add a significant error to a reading by the rounding process. Of course, it is silly to carry an unnecessarily large number of significant figures (sig. figs.), but it is not a crime. It is a more serious offence if, after spending a lot of effort to measure something, you make the result less accurate by rounding.

What is a good rule to follow? Typically one doesn't know the size of the error to better than about 10%. Therefore, one generally has to keep two significant figures to express the error - this may seem excessive, but consider the following. If we round a number like 9.4% to 9% (we are now talking about the error) we are making about a 10% "error in the error" and quoting 9% implies that the error is somewhere between 8.5 and 9.5% - this is fine. However, if the error is 1.1% say, and we round off to 1%, then the implied range is 0.5 to 1.5%, a huge range - this is not fine.

A very safe "rule of thumb" is the following: assume that errors are rarely known to better than about 10%. Then, when you round the value of your physical measurement, make sure that the rounding does not add an error that is larger than 10% of your estimated error.

7.1 An example

Suppose that you have a reading $y = 5.76573$ where your estimated error is 0.012. Then a reasonable procedure is to quote $y = 5.765 \pm 0.012$. If you had rounded to 5.77, then you would be adding an error that is comparable to the measurement error (because the roundoff would be $5.77 - 5.76573 = 0.0043$, which is over 30% of the estimated error of 0.012).

8 Linearizing Data

We used nonlinear regression in order to find the resonance frequency ω_0 and the FWHM γ for the LCR resonance curve. This is easy enough to do with Mathematica, but linearizing data can often give you a better understanding of the fit-parameters and can be done by hand. This approach does not work with all types of mathematical relationships, but does work for a group of important functions, namely: exponentials, power-laws and logarithmic dependencies:

For exponentially related quantities, plot $ln(y)$ vs t: For $y(t) = y_0 e^{-t/\tau}$, plot $ln(y)$ vs. t and get $\hat{y} = ln(y) = ln(y_0) - t/\tau$. You then plot $\hat{y}$ vs t and you can extract y_0 from the y-intercept as $y_0 = exp$(y-intercept) and $\tau = -1/$slope.

For power-law relationships, plot $ln(y)$ vs $ln(x)$: We now have a function of the type $y(x) = ax^n$, so by plotting $ln(y)$ vs $ln(x)$ we should get a straight line in the following manner: $\hat{y} = ln(y) = ln(a) + nln(x) = ln(a) + n\hat{x}$. Now, plot $\hat{y}$ vs $\hat{x}$ and get the quantity $a = exp$(y-intercept) and the exponent $n =$ slope.

Finally, for logarithmic relationships, plot y vs $ln(x)$: In the case of a logarithmic relationship, $y(x) = aln(x)$, you plot y vs $ln(x)$, and the slope equals a.