Physics 501
Normal Modes Summary

We start with a fairly general Lagrangian for N quadratic degrees of freedom:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{N} \dot{x}_i M_{ij} \dot{x}_j \; - \; \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \; .$$

Both M and K are real and symmetric matrices ($M_{ij} = M_{ji}$) and to enable us to use matrix notation, we will think of x as a single column matrix. i and j will always run from 1 to N and will always label matrix/vector components. So, for example,

$$(Mx)_i = \sum_{j=1}^{N} x_j M_{ji} = \sum x_j M_{ij} \; .$$

With this notation in hand, we can now write down the equations of motion

$$-M\ddot{x} = Kx \; .$$

There must exist a complete basis of normal mode solutions $x(t) = Ae^{i\omega t}$ where we will define $A = \sqrt{M}v$ so that the equation of motion becomes

$$\omega^2 \; v = \left(\frac{1}{\sqrt{M}} K \frac{1}{\sqrt{M}} \right) \; v \equiv \frac{K}{M} \; v \; .$$

This is an eigenvalue problem, and since $\frac{K}{M}$ is a symmetric matrix (as you can easily check yourself), there exists a complete, orthonormal set of eigenvectors $v_{(b)}$. I will put brackets around the indices which label different vectors as opposed components of the same vector. The index b runs from 1 to N. We have the following properties:

$$\frac{K}{M} \; v_{(b)} = (\omega_b)^2 \; v_{(b)} \; ,$$

$$\sum_b (v_{(b)})_i (v_{(b)})_j = \delta_{ij} \qquad \text{(completeness)},$$

$$\sum_i (v_{(b)})_i (v_{(c)})_i = \delta_{bc} \qquad \text{(orthonormality)}.$$

With these facts in hand, we can now proceed to quantize the system. The canonical momentum to x_i is

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = (M\dot{x})_i$$

and the Hamiltonian is

$$H = \frac{1}{2}\sum_{i,j=1}^{N} p_i \left(\frac{1}{M}\right)_{ij} p_j \; + \; \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \; .$$

Canonical quantization demands

$$[x_i, p_j] = i\hbar\delta_{ij} \ .$$

In analogy to the single harmonic oscillator, we define raising and lowering operators for each normal mode

$$a_b^\dagger = \sqrt{\frac{\omega_b}{2\hbar}} \sum_i \left(\sqrt{M} v_{(b)}\right)_i x_i \ - \ i \sqrt{\frac{1}{2\hbar\omega_b}} \sum_i \left(\frac{1}{\sqrt{M}} v_{(b)}\right)_i p_i \ ,$$

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As we saw in class, this leads to

$$[a_b, a_c^\dagger] = \delta_{bc}$$

$$[a_b, a_c] = 0$$

$$[a_b^\dagger, a_c^\dagger] = 0$$

and

$$H = \sum_{b=1}^{N} \hbar\omega_b \left(a_b^\dagger a_b + \frac{1}{2}\right) \ .$$

Therefore, the system has decomposed into N independent HOs. A complete basis of states can be labeled by giving the occupation number of each oscillator, n_b, corresponding to the eigenvalue of the number operator $N_b \equiv a_b^\dagger a_b$ (these ns are non-negative integers). The energy of such a state is

$$E = \sum_{b=1}^{N} \hbar\omega_b \left(n_b + \frac{1}{2}\right) \ .$$