Physics 501
Coherent State Summary

In what follows, the the Baker-Hausdorff formula

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \frac{1}{6}[X,[X,[X,Y]]] + \ldots + \frac{1}{n!}[X,[X,[X\ldots[X,Y]\ldots]]] + \ldots$$

will come in handy, as will the Baker-Campbell-Hausdorff formula (the lowest order)

$$\text{If } [X,Y] \text{ commutes with both } X \text{ and } Y \text{ then } e^X e^Y = e^{X+Y+[X,Y]/2}$$

We will work with a single Harmonic Oscillator mode, with Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{k}{2}x^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \qquad \omega = \sqrt{\frac{k}{m}}$$

where the relationship between x and p and the lowering and raising operators is

$$x = \sqrt{\frac{2\hbar}{m\omega}}\left(\frac{a + a^\dagger}{2}\right)$$

$$p = \sqrt{2\hbar m\omega}\left(\frac{a - a^\dagger}{2i}\right)$$

It will be useful to define the normalized position and momentum,

$$X = \sqrt{m\omega}\, x = \sqrt{2\hbar}\,\frac{a + a^\dagger}{2}$$

and

$$P = \sqrt{\frac{1}{m\omega}}\, p = \sqrt{2\hbar}\,\frac{a - a^\dagger}{2i}$$

We also have the energy eigenstates

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle$$

where the vacuum state is defined by

$$a|0\rangle = 0$$

Having reviewed the notation, we can now define the coherent states with

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

and

$$\langle\alpha|\alpha\rangle = 1$$

One important property is that

$$\langle\alpha|a|\alpha\rangle = \alpha \qquad \langle\alpha|a^\dagger|\alpha\rangle = \bar{\alpha}$$

$$\langle\alpha|X|\alpha\rangle = \sqrt{\frac{\hbar}{2}}\,\Re(\alpha) \qquad \langle\alpha|P|\alpha\rangle = \sqrt{\frac{\hbar}{2}}\,\Im(\alpha)$$

The coherent states can be defined explicitly using a translation operator in the X-P:

$$D(\alpha) \equiv e^{\alpha a^\dagger - \bar{\alpha}a}$$

This operator is unitary, and $D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha)$ and when acting on a and $a^\dagger$, it affects a shift:

$$D^{-1}(\alpha)aD(\alpha) = a + \alpha \qquad D^{-1}(\alpha)a^\dagger D(\alpha) = a^\dagger + \bar{\alpha}$$

It is now easy to show that $|\alpha\rangle = D(\alpha)|0\rangle$ has the right properties.

One can also show that in a coherent state $|\alpha\rangle$

$$\langle(\Delta X)^2\ \rangle = \langle(\Delta P)^2\ \rangle = \frac{\hbar}{2}$$

implying that the state has minimal uncertainty and is balanced. This is the closest we can get to a state with a well defined X and P.

In the Schrodinger picture, under the action of the Hamiltonian, the time evolution of the coherent states can be shown to be

$$|\alpha, t\rangle \equiv e^{-iHt/\hbar}|\alpha\rangle = e^{i\omega t/2}|e^{i\omega t}\alpha\rangle$$

which means that the coherent states evolve in the X-P plane by rotation.

The average excitation number is

$$\langle n\rangle = \langle\alpha|n|\alpha\rangle = |\alpha|^2$$

The coherent state has a Poisson distribution in the excitation number:

$$|\langle m|\alpha\rangle|^2 = \frac{\langle n\rangle^m e^{-\langle n\rangle}}{m!}$$

As usual with the Poisson distribution, the width squared is equal to the mean:

$$\langle(\Delta n)^2\rangle = \langle n\rangle$$

and the distribution gets narrower for large $\langle n \rangle$ in the following sense:

$$\langle n \rangle \;\to\; \infty \quad \Rightarrow \quad \frac{\sqrt{\langle (\Delta n)^2 \rangle}}{\langle n \rangle} \;\to\; 0$$

The coherent states are over-complete. This meant that while they form a complete set:

$$\int d\alpha^2 \frac{|\alpha\rangle\langle\alpha|}{\pi} = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1$$

there are too many of them, and their overlap is nonzero:

$$|\langle \alpha' | \alpha \rangle| = e^{-|\alpha' - \alpha|^2/2}$$

The coherent states have also the following very nice property, which makes them useful for theoretical calculations. Let $F(a, a^\dagger)$ be any operator made up of as and $a^\dagger$s, and let it have a normal ordered form like this: $F(a, a^\dagger) = \sum_i f_i(a^\dagger) g_i(a)$. Then

$$\langle \alpha | F(a, a^\dagger) | \alpha \rangle = \sum_i f_i(\bar{\alpha}) g_i(\alpha)$$

Let's get back to photons now. If we focus on one normal mode (with a well defined momentum vector and polarization, and therefore a fixed frequency), the classical plane wave radiation is best approximated in quantum mechanics with a coherent state in that particular mode. This coherent state has maximally well defined electric and magnetic fields (which are equivalent to x and p), and these fields evolve harmonically, as expected. The number of photons in not well defined, but its mean is proportional to the total intensity of the pulse. With a large number of photons, the fractional fluctuation in photon number is small.

This kind of coherent state radiation is what you get in a continuous beam (as opposed to pulsed), high quality, quiet laser. The (unavoidable) fluctuations in photon number limit the precision with which measurements can be made with such a beam. This is known as the quantum limit. Where really high precision is required (in gravity wave detectors, for example), squeezed light can be used to go beyond this limit.

Since any vacuum electromagnetic field configuration can be decomposed into plane wave solutions, any classical electromagnetic field can be thought of as build from coherent states. In Quantum Field Theory, coherent states are used often to describe classical field profiles.