Physics 501 Problem Set 6

Due date to be annouced - I need to check with the TA.

1. Path integrals: the propagator for the HO.

(a) Let M by a symmetric, real, $n \times n$ matrix with eigenvalues λ_k. Prove that

$$\int \prod_{k=1}^{n} dy_k \; \exp\left(-\sum_{i,j} y_i M_{ij} y_j\right) = \prod_{k=1}^{n} \left(\frac{\pi}{\lambda_k}\right)^{1/2} = \frac{\pi^{n/2}}{\sqrt{\det M}}$$

This is a handy little formula. Hint: diagonalize M by changing the variables of integration.

(b) We want everything to converge nicely, so we will start the calculation in the euclidean formalism, where $L_E = m\dot{x}^2/2 + kx^2/2$ and our oscillator has frequency $\omega^2 = k/m$. Let's first compute the propagator with both endpoints at 0: $K_E(x_b = 0, T; x_a = 0, 0)$. Following our approach in class, discretise the time interval T into N+1 small intervals, and define $\epsilon = T/(N+1)$. Argue that you need to compute the following gaussian integral:

$$K_E(x_b = 0, T; x_a = 0, 0) = \lim_{N\to\infty} \left(\frac{m}{2\pi\epsilon}\right)^{(N+1)/2} \int \prod_{n=1}^{N} dx_n$$

$$\exp\left[-\frac{m}{2\epsilon}\left((x_N)^2 + (x_N - x_{N-1})^2 + \ldots + (x_3 - x_2)^2 + (x_2 - x_1)^2 + (x_1)^2\right)\right] \times$$

$$\exp\left[-\frac{k\epsilon}{2}\left((x_N)^2 + (x_{N-1})^2 + \ldots + (x_3)^2 + (x_2)^2 + (x_1)^2\right)\right]$$

and that the answer, using part (a) can be written as

$$K_E(x_b = 0, T; x_a = 0, 0) = \lim_{N\to\infty} \left(\frac{m}{2\pi\epsilon}\right)^{(N+1)/2} \frac{\pi^{N/2}}{\sqrt{\det A}}$$

where A is an $N \times N$ matrix with entries

$$A_{ij} = \left(\frac{m}{\epsilon} + \frac{k\epsilon}{2}\right)\delta_{ij} - \frac{m}{2\epsilon}\;\delta_{i,j\pm1}$$

.

(c) BONUS This part is for bonus marks. I will give out no hints. Hint: this question is not really all that hard.

Show that

$$\lim_{N\to\infty} \left(\frac{2\epsilon}{m}\right)^{N+1} \det A = \frac{2}{\sqrt{mk}} \sinh(\omega T)$$

(d) Put it all together (including the result in part (c)) to show that

$$K_E(x_b = 0, T; x_a = 0, 0) = \sqrt{\frac{m\omega}{2\pi\sinh(\omega T)}}$$

and then extract the ground state energy of the harmonic oscillator from the large T behaviour of K.

(e) As we saw in class, to extend the result to different starting and ending points, all we need to do is compute the classical action. Let's move back to Lorentzian time. The most general classical path is $x(t) = \gamma \cos(\omega t + \delta)$. Taking γ and δ such that $x(0) = x_a$ and $x(\Delta t) = x_b$, show that the classical action is

$$S_{cl}(x_b, \Delta t; x_a, 0) = \frac{m\omega}{2\sin(\omega \Delta t)} \left[(x_a^2 + x_b^2)\cos(\omega \Delta t) - 2x_a x_b \right]$$

(f) Analitically continuing (d) to Lorentzian time, and putting it together with (e), argue that the complete propagator is

$$K(x_b, \Delta t; x_a, 0) = \sqrt{\frac{m\omega}{2\pi i \sin(\omega \Delta t)}} \exp\left(\frac{im\omega}{2\sin(\omega \Delta t)} \left[(x_a^2 + x_b^2)\cos(\omega \Delta t) - 2x_a x_b \right] \right)$$

(g) Take the limit as $\omega \to 0$ and recover the free particle propagator we have derived in class.

2. The Dirac monopole.

Confirm that the following vector potentials A

$$\vec{A}^I = e_M \frac{1 - \cos\theta}{r \sin\theta} \hat{\phi}$$

$$\vec{A}^{II} = e_M \frac{-1 - \cos\theta}{r \sin\theta} \hat{\phi}$$

lead to the same vector field $B = (e_M/r^2)\hat{r}$ and that the gauge transformation which connects them ($A^{II} - A^I = \nabla \Lambda$) is given by

$$\Lambda = -2e_M \phi$$

3. The Aharonov-Bohm grating.

Take a grating with spacing d and a screen a distance D away, as shown. There is a long, impenetrable solenoid between any two adjacent slits. Each solenoid carries a magnetic flux Φ.

Electrons with wavenumber k impinge on the grating. Calculate the spacing of the diffraction fringes and the shift of the diffraction pattern per unit magnetic flux.