ASTR 530

# Essential Astrophysics

Course Notes

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# 1 Introduction and review

Several text books present an overview of astrophysics at the advanced undergraduate or introductory graduate level. You might like to refer to:

1. Longair, *High Energy Astronpysics*, Cambridge University Press, ISBN 978-0-521-75618-1 (2011)
2. Irwin, *Astrophysics*, Wiley, ISBN 978-0-470-01306-9 (2007)
3. Ryden and Peterson, *Foundations of Astrophysics*, Addison-Wesley, ISBN 978-0-321-59558-4 (2009)
4. Bowers and Deeming, Astrophysics II and II, Jones and Bartlett, ISBN 0-86720-018-9 (1984)
5. Rybicki and Lightman, Radiative processes in Astrophysics, Wiley, ISBN 978-0-471-82759-7 (1979)

## 1.1 Constants and conversion factors

Generally, it is preferable to use SI units when calculating quantities. However, these can be cumbersome and astronomers often employ units that are more appropriate to the application. Some typical astronomy units are listed in Table 1.1. For reference, some useful physical constants are listed in Table 1.2.

Table 1.1: Common units used in astronomy

| Name | Symbol | SI value |
|---|---|---|
| astronomical unit | AU | $1.4960 \times 10^{11}$ m |
| parsec | pc | $3.0857 \times 10^{16}$ m |
| light year | ly | $9.4607 \times 10^{15}$ m |
| solar radius | $R_\odot$ | $6.955 \times 10^{8}$ m |
| Earth radius | $R_\oplus$ | 6371 km |
| year (Julian) | y | $3.15576 \times 10^{7}$ s |
| arcsecond | arcsec | $4.84814 \times 10^{-6}$ rad |
| solar mass | $M_\odot$ | $1.98855 \times 10^{30}$ kg |
| Earth mass | $M_\oplus$ | $5.97219 \times 10^{24}$ kg |
| electron volt | eV | $1.60218 \times 10^{-19}$ J |
| gauss | g | $10^{-4}$ T |
| Jansky | Jy | $10^{-26}$ W m$^{-2}$ Hz$^{-1}$ |

Table 1.2: Physical constants

| Name | Symbol | SI value |
|---|---|---|
| speed of light in vacuum | $c$ | $3.99792 \times 10^{8}$ m/s |
| Gravitational constant | $G$ | $6.67384 \times 10^{-11}$ M m$^2$ kg$^{-1}$ |
| Planck constant | $h$ | $6.62607 \times 10^{-34}$ J s |
| reduced Planck constant | $\hbar$ | $1.05457 \times 10^{-34}$ J s |
| Boltzmann's constant | $k$ | $1.38065 \times 10^{-23}$ J K$^{-1}$ |
| permittivity of free space | $\varepsilon_0$ | $8.85419 \times 10^{-12}$ C$^2$ J$^{-1}$ m$^{-1}$ |
| permeability of free space | $\mu_0 = 1/\varepsilon_0 c^2$ | $4\pi \times 10^{-7}$ N A$^{-2}$ |
| electron charge | $e$ | $1.6021765 \times 10^{-19}$ C |
| fine structure constant | $\alpha = e^2/4\pi\varepsilon_0\hbar c$ | $7.29735 \times 10^{-3}$ |
| electron mass | $m_e$ | $9.10938 \times 10^{-31}$ kg |
| proton mass | $m_p$ | $1.67262 \times 10^{-27}$ kg |
| atomic mass unit | amu | $1.66054 \times 10^{-27}$ kg |
| hydrogen mass | $m_H$ | $1.67382 \times 10^{-27}$ kg |
| Bohr magneton | $\mu_B = eh/4\pi m_e c$ | $9.274 \times 10^{-24}$ J T$^{-1}$ |
| radiation constant | $a = 8\pi^5 k^4/15c^3h^3$ | $7.56572 \times 10^{-16}$ J m$^{-3}$ K$^{-4}$ |
| Stefan-Boltzmann constant | $\sigma = ac/4$ | $5.67037 \times 10^{-8}$ W m$^{-e}$ K$^{-1}$ |

## 1.2 Other systems of units

Gaussian units are commonly used in electromagnetic equations. In these units, quantities such as electric and magnetic fields have different relationships to each other than in SI units. For example, in Gaussian units, $E$, $D$, $B$, $H$, $P$, and $M$ all have the same units, while in SI they are different. It is important to be familiar with both systems. Table 1.3 gives a summary of the most relevant relations.

In high-energy physics and cosmology natural units are commonly employed. In this system, one sets many fundamental constants to unity. For example, in units in which $c = 1$, time and distance have the same units, metres for example (a metre of time is the length of time required for light to travel a distance of one metre in vacuum). Energy and mass also have the same units. If $\hbar = 1$, energy has units of reciprocal time. If $k = 1$, temperature has the same units as energy. Typically one chooses, $c = \hbar = k = \varepsilon_0 = 1$. It is always possible to convert the resulting equations to SI units by using dimensional analysis to insert the "missing" constants.

A fundamental unit of mass, the *Planck mass*, can be formed from $G$, $\hbar$, and $c$,

$$m_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.17651 \times 10^{-8} \text{ kg} \tag{1.1}$$

For this mass, the Compton radius $\lambda_c = h/mc$ is about equal to the Schwarzchild radius $r_S = 2Gm/c^2$.

It is also possible to set $G = 1$, in which case $m_P = 1$ and masses are dimensionless numbers, which give the mass in units of the Planck mass. Similarly, distances are then in units of the Planck length and time is in units of the Planck time. More commonly, $G = m_P^{-2}$ is not set to zero but is regarded as a *coupling constant*, describing the strength of the gravitational force.

Table 1.3: Comparison between electromagnetic equations in SI, Gaussian and natural units

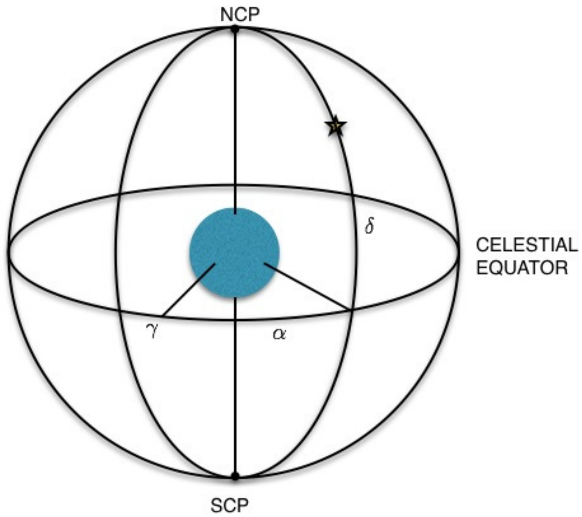
| Name | SI | Gaussian | Natural |
|---|---|---|---|
| Coulomb's law | $F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2}$ | $F = \frac{q^2}{r^2}$ | $F = \frac{1}{4\pi}\frac{q^2}{r^2}$ |
| | $\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$ | $\boldsymbol{E} = \frac{q}{r^2}\hat{\boldsymbol{r}}$ | $\boldsymbol{E} = \frac{1}{4\pi}\frac{q}{r^2}\hat{\boldsymbol{r}}$ |
| Biot-Savart law | $\boldsymbol{B} = \frac{\mu_0}{4\pi}\oint\frac{Id\boldsymbol{l}\times\hat{\boldsymbol{r}}}{r^2}$ | $\boldsymbol{B} = \frac{1}{c}\oint\frac{Id\boldsymbol{l}\times\hat{\boldsymbol{r}}}{r^2}$ | $\boldsymbol{B} = \frac{1}{4\pi}\oint\frac{Id\boldsymbol{l}\times\hat{\boldsymbol{r}}}{r^2}$ |
| Lorentz force | $\boldsymbol{F} = q\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)$ | $\boldsymbol{F} = q\left(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)$ | $\boldsymbol{F} = q\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)$ |
| Energy density and flux | $U = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$ | $U = \frac{1}{8\pi}\left(E^2+B^2\right)$ | $U = \frac{1}{2}\left(E^2+B^2\right)$ |
| | $\boldsymbol{S} = \frac{1}{\mu_0}\left(\boldsymbol{E}\times\boldsymbol{B}\right)$ | $\boldsymbol{S} = \frac{1}{4\pi c}\left(\boldsymbol{E}\times\boldsymbol{B}\right)$ | $\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{B}$ |
| Vector and scalar potentials | $\boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}$ | $\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}$ | $\boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}$ |
| | $\boldsymbol{B} = -\nabla\times\boldsymbol{A}$ | $\boldsymbol{B} = -\nabla\times\boldsymbol{A}$ | $\boldsymbol{B} = -\nabla\times\boldsymbol{A}$ |
| Vacuum Maxwell equations | $\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ | $\nabla\cdot\boldsymbol{E} = 4\pi\rho$ | $\nabla\cdot\boldsymbol{E} = \rho$ |
| | $\nabla\cdot\boldsymbol{B} = 0$ | $\nabla\cdot\boldsymbol{B} = 0$ | $\nabla\cdot\boldsymbol{B} = 0$ |
| | $\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$ | $\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}$ | $\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$ |
| | $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}$ | $\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$ | $\nabla\times\boldsymbol{B} = \boldsymbol{J} + \frac{\partial\boldsymbol{E}}{\partial t}$ |

## 1.3 Celestial coordinates

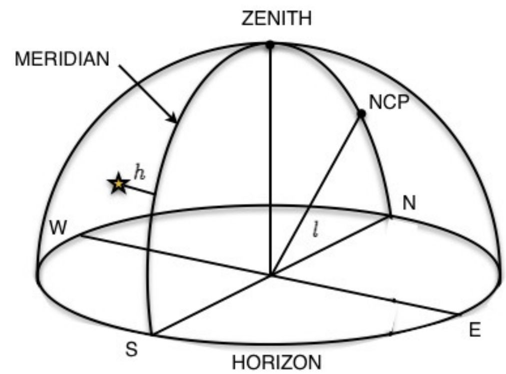
Astronomers often need to refer to the positions of things in the sky. This is most conveniently done using spherical polar coordinates. One must choose a location for the origin, and the orientation. Several systems are in use. The most common is the celestial coordinate system that has its origin at the centre of the Earth. The $z$ axis is aligned with the Earth's spin axis, with the positive direction being North. It is useful to imagine the sky as being a large sphere, the *celestial sphere*, centred on the Earth. The Earth's axis intersects the celestial sphere at the North and South *celestial poles* (NCP and SCP respectively). The projection of the Earth's equator onto the celestial sphere is a great circle called the *celestial equator*.

Rather than using the polar angle $\theta$, measured from the NCP, it is more common to use *declination* $\delta$, which is the angle measured from the celestial equator. Declination is positive in the northern hemisphere and negative in the southern hemisphere. Obviously, $\delta = \pi/2 - \theta$.

The azimuth angle $\phi$ is called *right ascension* and is denoted by the symbol $\alpha$. It is normally measured in hours, minutes and seconds rather than degrees (24 hours = 360 degrees). The zero point of right ascension is the point on the celestial sphere where the Sun crosses the celestial




(a)




(b)

Figure 1.1: (a) Celestial coordinates are a non-rotating spherical coordinate system centred on the Earth. Right ascension $\alpha$ is measured from the First Point of Aries $\gamma$ and declination $\delta$ is measured from the celestial equator. (b) The sky as seen by an observer at latitude $l$. A star is shown that has just passed the meridian, which now has positive hour angle $h$.

equator from South to North. This point is called the *first point of Aries* and given the symbel $\gamma$. The Sun passes it on the *vernal equinox* (around March 20).

A problem with this system is that the Earth's axis precesses, due to the action of lunar and solar tidal forces. This precession has a period of about 25,772 years and results in a slow but significant change in the right ascension and declination of celestial objects. Therefore, when giving the coordinates of an object it is necessary also to indicate the time, or *epoch* at which they are valid. The epoch commonly used today is 12 noon on January 1, 2000 (universal time), which is called J2000. Formulae for precession of coordinates from one epoch to another are given by J. Meeus, "Astronomical Algorithms" (1981).

When computing the positions of celestial objects, it is also necessary to correct for aberration - an effect due to the motion of the Earth and the finite speed of light, and nutation - a nodding motion of the Earth's axis due to the tidal forces of the Sun and Moon. Each effect can be as large as $\sim 20$ arcsec.

## 1.4 Spherical triangles

We often encounter triangles on a sphere when using celestial coordinates. For example, we may need to know how far an object is from the zenith, for a given hour angle and declination. This is easily computed using spherical triangles. A spherical triangle has sides formed by geodesics, as in Fig 1.2. Let the arc length of the sides (the angle subtended at the centre of the sphere) be $a$, $b$, and $c$. Let the interior angles, measured on the surface of the sphere, at the vertices be $A$, $B$, and $C$, where $A$ is the vertex opposite side $a$, etc. Then, the *spherical sine rule* is

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \tag{1.2}$$

and the *spherical cosine rule* is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{1.3}$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \tag{1.4}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{1.5}$$

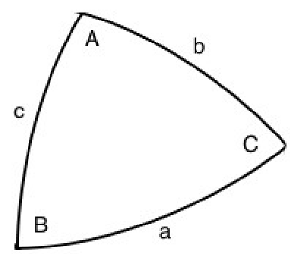



Figure 1.2: A spherical triangle is a triangle formed by intersecting geodesics on the surface of a sphere.

As an example, the zenith angle $\zeta$ of an object at hour angle $h$ and declination $\delta$, seen from latitude $l$ is given by

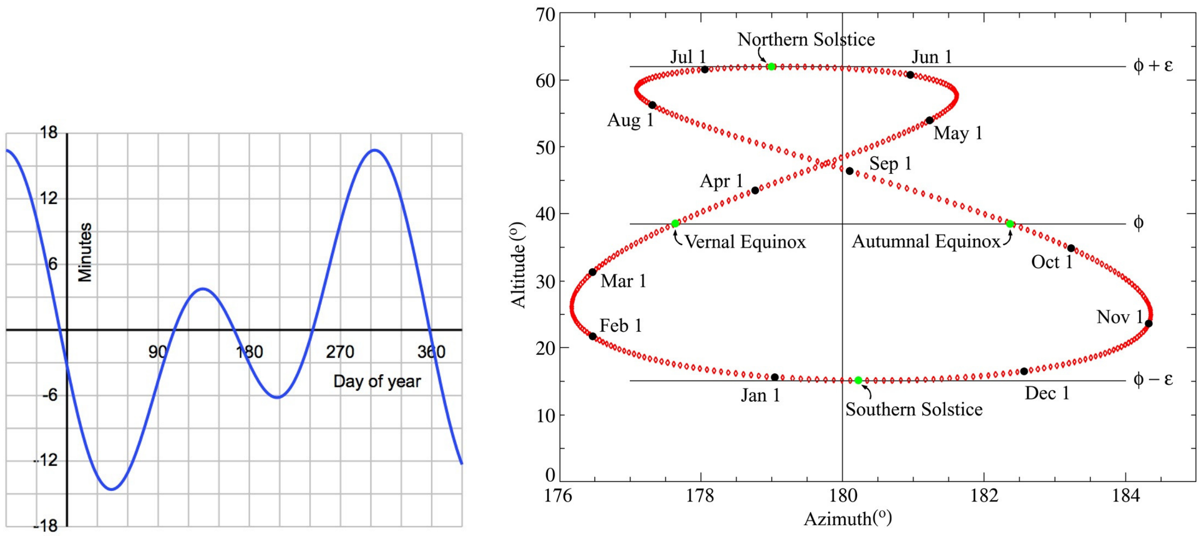
$$\cos \zeta = \sin \delta \sin l + \cos \delta \cos l \cos h. \tag{1.6}$$

## 1.5 Time

Historically, time was based on the rotation of the Earth. The time interval between successive transits (crossing of the meridian) of the Sun is the *solar day*. Subdividing this gives *apparent solar time*. However because of the tilt of the Earth's axis, and eccentricity of its orbit, this time interval is not constant. If we average it over a year, we get *mean solar time*. The difference between apparent solar time and mean solar time is the *equation of time* (Figure 1.3). Over a year, the position of the Sun is the sky at noon, mean solar time, traces out a vertical figure eight called the *analemma*.

The time measured with respect to the Sun at any given place is called *local time*. Since it depends on position, we define *universal* or "Greenwich" time as the local time at 0° longitude (the *prime meridian*). To convert local time to universal time, subtract the longitude divided by 15 (to convert degrees to hours).

The universal time defined by the position of distant quasars, as measured by a fixed observatory on Earth, is called *UT0*. However, it is affected by variations in the orientation of the Earth's spin




(a) (b)

Figure 1.3: (a) Equation of time. The graph shows the difference between mean solar time, read by a watch, and apparent solar time, read from a sundial. (Drini/Zazou, Wikimedia Commons). (b) The solar anelemma. (JPL Horizons, Wikimedia Commons).

axis (polar wandering). Correcting for this effect gives *UT1*, which typically differs from UT0 by a few ms. UT1 is used by telescope control systems when pointing to celestial objects, and for determining astrometric positions.

A further problem arises because the Earth's rotation rate is not constant. Lunar tidal torques are slowing the Earth's rotation, transferring angular momentum to the Moon's orbit. Because of this, the mean solar day is increasing at a rate of about 1.7 ms per century. Universal Time Coordinated (UTC) is defined by atomic clocks. As the Earth spins down, UT1 falls behind UTC. The difference is kept to within a second by adding *leap seconds* to UTC as needed. UTC is the time used for physics experiments.

*Sidereal time* differs from mean solar time, UT1 and UTC in that it refers to the positions of stars (or quasars) and not the Sun. A sidereal day is the time between successive transits of stars. It is shorter than a solar day by about 4 minutes (stars rise about 4 minutes earlier each night). To be precise, sidereal time is the right ascension of the meridian, which of course increases as the Earth rotates. For example, if the sidereal time is 10 hrs, a star whose right ascension is 10 hrs would just be crossing the meridian (and therefore at its highest position in the sky). Sidereal time is useful in determining when to observe objects. The sidereal time at midnight local time is 12.0 hours at the vernal equinox and increases by two hours each month.

Astronomers generally use *Julian dates* to represent times of observations. Unlike calendar dates, Julian days begin at noon, and are not affected by leap years. The *Julian Day Number* (JDN) is an integer giving continuous count of days since *noon* on January 1, 4713 BC. The day starting at noon on January 1, 2000, was JDN 2,451,545. The *Julian Date* (JD) is the JDN for the day beginning at noon UTC, plus a decimal fraction representing the fraction of the day since that time.

A *Julian year* is defined to be exactly 365.25 days. Similarly, a *Julian Century* is exactly 36525 days. The slowing of the Earth's rotation implies that Julian days are slowly increasing in length. Standard algorithms are available to convert between Julian days and the Gregorian calendar.

The *Reduced Julian Date* equals JD − 2400000 and the *Modified Julian Date* (MJD) equals JD − 240000.5. There is also a *Truncated Julian Date* that equals JD − 2440000.5. UNIX time (the number of seconds since midnight January 1, 1970) equals (JD − 2440587.5) × 86400.

# 2 Solar System

The Solar System consists of the Sun and four rocky *terrestrial* planets (Mercury, Venus, Earth and Mars) and four gas giant *Jovian* planets (Jupiter, Saturn, Uranus and Neptune). There are also numerous smaller objects including minor planets (e.g. Pluto), trans-Neptunian objects, asteroids and comets. In addition, we have interplanetary dust, solar plasma and magnetic fields. A review of the properties of the Solar System is beyond the scope of this course. The topic is covered in all introductory astronomy texts and may specialized books. Here we shall just mention a few points that have general application.

We observe the Universe from a moving platform. The Earth orbits the Sun with a speed of approximately 30 km/s. The resulting Doppler shift affects both the radial velocities measured for stars and also the timing of radio signals received from pulsars. Fortunately, it is easy to correct for these effects since the Earths orbital parameters are accurately known. Normally, velocities, positions and times for celestial objects are given in the *heliocentric* inertial coordinate system.

## 2.1 Planetary motion

To first order, the motion of planets, and smaller objects, in the solar system is described by Kepler's three laws of planetary motion:

1. The shape of a planet's orbit is an ellipse with the Sun at a focus.
2. A line connecting the planet and the Sun sweeps out equal areas in equal times.
3. The square of the orbital period is proportional to the cube of the semi-major axis of the orbit.

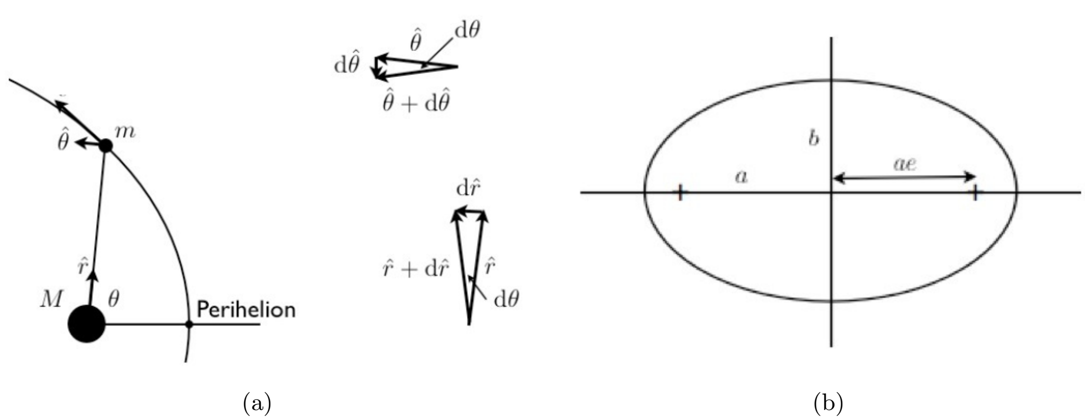



Figure 2.1: (a) Geometry of the Kepler problem. (b) Properties of an ellipse.

It is straightforward to show that these laws follow directly from Newton's mechanics and law of gravity. Since gravity is a central force, no torque acts on the planet so angular momentum is conserved. The orbit is therefore confined to a plane perpendicular to the angular momentum vector $\boldsymbol{L}$. Chose a polar coordinate system $(r, \theta)$ in this plane, with the Sun at the origin and let $\boldsymbol{r}$ be a vector connecting the Sun to the planet. $\hat{\boldsymbol{r}}$ is a unit vector in the $r$ direction and $\hat{\boldsymbol{\theta}}$ is an orthogonal unit vector. Denoting a time derivative by a dot, Newton's laws give

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^2}\hat{\boldsymbol{r}}. \tag{2.1}$$

Expanding the LHS, noting that $\dot{\hat{\boldsymbol{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}$ and $\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}}$, and collecting coefficients of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$, we get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}, \tag{2.2}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \tag{2.3}$$

The second equation is easily integrated to give

$$L = r^2\dot{\theta} \tag{2.4}$$

where the constant $L$ is the orbital angular momentum of the planet, per unit mass.

To get an equation for the orbit, we need to eliminate time. Rearranging (2.4) gives

$$\begin{aligned}\frac{d}{dt} &= \dot{\theta}\frac{d}{d\theta} \\ &= \frac{L}{r^2}\frac{d}{d\theta}.\end{aligned} \tag{2.5}$$

Putting this into (2.2) gives

$$\frac{d}{d\theta}\frac{1}{r^2}\frac{dr}{d\theta} - \frac{1}{r} = -\frac{GM}{L^2}. \tag{2.6}$$

This can be simplified using the substitution $u = 1/r$, which gives a harmonic oscillator equation

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{L^2}. \tag{2.7}$$

The general solution is

$$u = A\cos\theta + B\sin\theta + \frac{GM}{L^2}, \tag{2.8}$$

where $A$ and $B$ are arbitrary constants. We are free to choose the direction corresponding to $\theta = 0$ and can use that freedom to set $B = 0$. Defining $\epsilon = AL^2/GM$ the solution becomes

$$r = \frac{L^2/GM}{1 + \epsilon\cos\theta}. \tag{2.9}$$

Comparing this with the equation of an ellipse of eccentricity $e$ and semi-major axis $a$,

$$r = \frac{a(1 - \epsilon^2)}{1 + \epsilon\cos\theta}, \tag{2.10}$$

we see that

$$\frac{L^2}{GM} = a(1 - \epsilon^2). \tag{2.11}$$

The point of closest approach (*perihelion*) occurs when $\theta = 0$, at a distance

$$r_p = a(1 - \epsilon) = \frac{L^2}{GM(1 + \epsilon)}. \tag{2.12}$$

Kepler's second law follows from conservation of orbital angular momentum and is therefore valid for any central force. An inverse square radial dependence of the force is required by Kepler's first law. Small deviations from this result in the orbit not closing, which causes a small precession of the perihelion (the point of minimum distance from the Sun). Gravitational perturbations from other planets cause Mercury's orbit to precesses by about 532 arcsec per Julian century. General relativity adds another 43 arcsec per century. Verification of this additional precession, in 1916, was an important test of GR.

Kepler's third law can be written more precisely in the form

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \tag{2.13}$$

where $P$ is the *sidereal period* of the orbit (in the celestial reference frame), $a$ is the semi-major axis and $m_1$ and $m_2$ are the masses of the Sun and planet. Both the Sun and the planet actually orbit about their common centre of mass. The two orbits have the same Period and eccentricity, but differ in their semimajor axis $a_1$ and $a_2$, where $a_1 + a_2 = a$.

Other useful formulae are the orbital energy per unit mass,

$$E = -\frac{GM_\odot}{2a} \tag{2.14}$$

The angular momentum per unit mass,

$$L = \sqrt{GM_\odot a(1 - e^2)} \tag{2.15}$$

and the orbital velocity,

$$v = \sqrt{GM_\odot \left(\frac{2}{r} - \frac{1}{a}\right)} \tag{2.16}$$

A number of celestial phenomena can be observed in the Solar System. A *conjunction* is the appearance of objects close to each other in the sky. An example is Venus, which appears closest to the Sun at *superior conjunction* (Venus behind the Sun) and *inferior conjunction* (Venus in front of the Sun). *Opposition* occurs when a planet is in the opposite direction from the Sun. For objects orbiting beyond the Earth, this generally means that they are near their minimum distance from Earth.

The time between successive oppositions of a planet, as seen from the Earth, is called the *synodic period*, $P_s$. It is related to the sidereal period $P$ by

$$\frac{1}{P_s} = \frac{1}{P_\oplus} - \frac{1}{P} \tag{2.17}$$

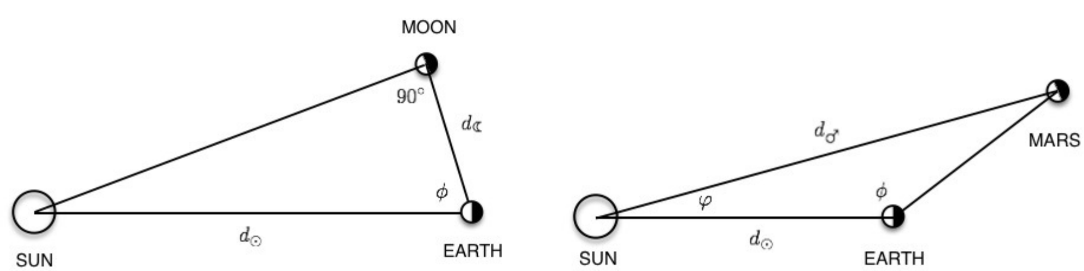
A *transit* occurs when a small object (in terms of angular size) passes in front of a larger object. For example during a transit of Mercury, one sees a small black disk moving across the Sun's face. If the foreground object has an angular size that is comparable to, or exceeds, that of the background object, we call this an *eclipse*. The Moon's angular size is sometimes smaller than that of the Sun, in which case we may see an *annular eclipse* and sometimes larger, which gives a *total eclipse*. The region of the Moon's shadow where the Sun it completely obscured is called the *umbra* and the region where it is partly obscured is called the *penumbra*. If the Moon passes through the Earth's shadow, we have a *lunar eclipse*.

If a distant object, having small angular size, passes behind a nearby object, we call this an *occultation*. An occultation of the radio source 3C48 by the Moon was instrumental in determining a more precise position for the source, allowing its optical identification as the first known quasar.

## 2.2 Distance measurements

Accurate distances are among the hardest things to measure in astronomy. For solar system objects, we might today use laser ranging, radar measurements, or timing of radio signals from space probes. Historically, distances have been measured by less direct means. The diameter of the Earth was first measured by Eratosthenese in 230 B.C., by measuring the angle of the shadow cast by the Sun at noon in Alexandria on the summer solstice. He knew that on that same day, the Sun would be directly overhead at Aswan (then called Syene) which is located on the Tropic of Cancer. Knowing the distance between Aswan and Alexandria, he could then estimate the circumference of the Earth. Once the diameter of the Earth is known, the distance to the Moon can be found by parallax. One can measure the apparent position of the moon at different times during the night. After correcting for the orbital motion, a difference remains that is due to the motion of the observer caused by Earth rotation.

The distance to the Sun can be estimated by measuring the elongation $\phi$ of the Moon (the angle between the Sun and Moon, as seen from the Earth when the Moon is exactly half illuminated (1/4 phase). From Figure 2.2 we see that $d_{\odot} = d_{☾} \sec\phi$. This gives an estimate of the *Astronomical Unit* (AU), which is the length of the semi-major axis of the Earth's orbit.




(a) (b)

Figure 2.2: (a) If the distance to the Moon is known, the distance to the Sun can be estimated from the elongation $\phi$ angle of the Moon at quarter phase. (b) Properties of an ellipse.

The distances to inner planets can now be estimated by measuring their maximum elongation angles, and assuming circular orbits.

The distances to outer planets can be found by measuring their elongation exactly one year after opposition. Referring to Figure 2.2b, the angle $\varphi = 2\pi/P$, where $P$ is the sidereal period of the planet in years. Using the sine rule,

$$d_{♂} = \frac{\sin\phi}{\sin(\varphi + \phi)} d_{\odot}. \tag{2.18}$$

Distances to nearby stars can be found by measuring their parallax, due to the orbital motion of the Earth. This is the basis for the definition of the parsec, which is the distance at which one astronomical unit subtends an angle of one arcsecond.

Beyond a few tens of parsecs, stellar parallax is increasingly hard to measure. We must rely on dynamical or statistical methods. Supernovae often produce an expanding shell of hot gas. The radius of the shell can be determined my multiplying the expansion velocity (from the Doppler shift of the spectral lines that it emits) by the time since the explosion. If the angular size can also be measured, the distance can then be found.

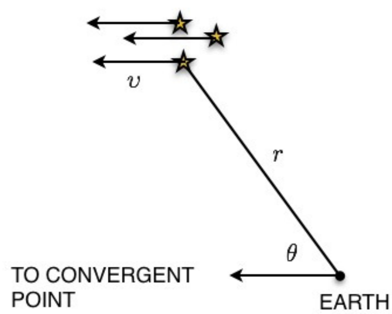



Figure 2.3: Moving cluster parallax. From the angle $\theta$ between the cluster and the apparent point of convergence, the proper motion and the radial velocity, one can estimate the distance to the cluster.

Another example is *moving cluster parallaxes*. The Hyades star cluster has a large angular extent on the sky, and is close enough that *proper motions* (angular velocity vectors) of its stars can be found by comparing images separated by a long time interval. These vectors are found to converge to a point several degrees away. By measuring the radial velocities of the stars (from the Doppler shift of their spectral lines) and using simple geometry, the distance to the cluster can be found. Referring to Figure 2.3, we see that radial and tangential velocity components are given by

$$v_r = v \sin\theta, \tag{2.19}$$

$$v_\perp = v \cos\theta, \tag{2.20}$$

where $\theta$ is the angle between the cluster and the point in the sky where the proper motion vectors converge. The magnitude of the proper motion is $\mu = v_\perp/r$ from which it follows that

$$r = v_r \tan\theta/\mu. \tag{2.21}$$

Such techniques extend the distance scale to hundreds of parsecs. At this point, the method of *standard candles* can be used. Essentially, one finds an object whose luminosity can be inferred and then measures its flux. The distance is then found from the inverse square law (3.14 in the next section). Standard candles within the Galaxy include RR-Lyrae stars and Cepheid variables. RR-Lyrae stars are pulsating stars whose luminosity lies within a reasonably-narrow range. Cepheid variables have a wider range of luminosities, but their luminosity is related to their pulsation period and can thus be calibrated.

With large telescopes, the distance scale can be extended to nearby galaxies by observing Cepheid's that they contain. At greater distances still, one can use the luminosities of the brightest stars, the sizes and luminosities of the largest HII regions (regions of ionized gas surrounding massive stars), or the luminosities of certain types of galaxies themselves. Obviously the distances become more and more uncertain as the number of steps in the "distance ladder" increases.

Recently, type Ia supernovae have been used to make reasonably-accurate distance estimates for distant galaxies. These are exploding stars, believed to result from matter accreting onto a white dwarf. One finds that the maximum luminosity of the explosion is related to the rate of decline of the light curve. Correcting for this gives luminosities with a scatter of less than 10%, resulting in distance estimates that are within 20% (Figure 2.4). Distances measured from type 1a supernovae led to the discovery that the expansion of the universe is accelerating, providing evidence for the existence of dark energy.

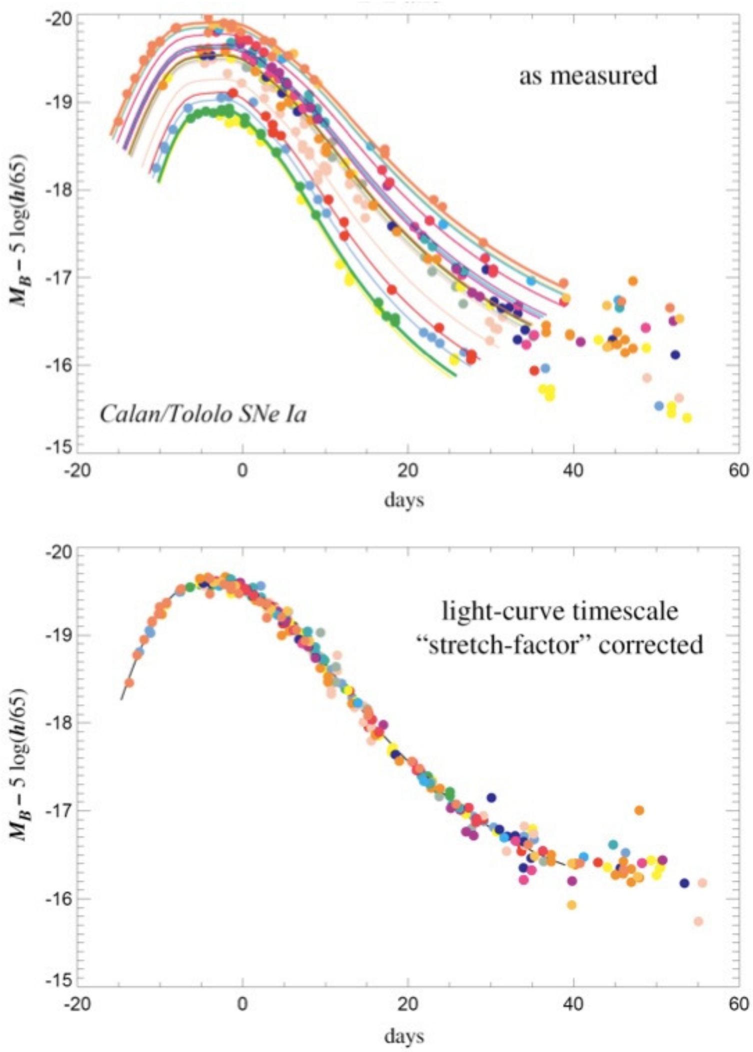



Figure 2.4: Light curves of Type 1a supernovae, before (upper panel) and after (lower panel) correction (Perlmutter and.Schmidt (2003) arXiv:astro-ph/0303428).

# 3 Photometric and astrometric measurements

Most of what we know about the Universe comes from observations of electromagnetic radiation. Telescopes are used to collect and detect the light, X-rays, infrared or radio waves. Detection is done using instruments that either respond to the energy of the radiation, or motion of electrons induced by the electromagnetic field. In all cases, the telescope system has an aperture, or effective area, which defines the amount of radiation that is intercepted. It also has some means of selecting the range of directions from which radiation can be detected, such as a collimator or imaging system.

The fundamental quantity describing the radiation field is the *specific intensity* $I_\nu(\boldsymbol{n})$, which is the radiant energy $E$ flowing in the direction $\boldsymbol{n}$ per unit time, per unit solid angle, per unit perpendicular area, per unit frequency interval.

$$I_\nu = \frac{dE}{dt\ d\Omega\ dS\ d\nu}. \tag{3.1}$$

The subscript $\nu$ reminds us that this quantity is a *spectral density.*

One can also define this in terms of angular frequency $I_\omega$. The power contained in interval $d\omega$, must equal the power contained within the corresponding frequency interval $d\nu = d\omega/2\pi$,

$$|I_\omega d\omega| = |I_\nu d\nu|, \tag{3.2}$$

therefore $I_\omega = I_\nu/2\pi$.

Similarly, one can define $I_\lambda$ wihich is the power per steradian per unit area per unit wavelength interval. Again, $|I_\lambda d\lambda| = |I_\nu d\nu|$, and since $\lambda\nu = c$ we have

$$I_\lambda = \frac{c}{\lambda^2} I_\nu. \tag{3.3}$$

To get the total power, per steradian, per unit area, we can integrate $I_\nu$ over all (positive) frequencies. This gives the *intensity.*

$$I = \int_0^\infty I_\nu d\nu. \tag{3.4}$$

I will often use the term *intensity* to refer either to the specific intensity, or the intensity itself. In any case, specific intensity will always be shown with the subscript, $\nu$ or $\lambda$.

If we average the intensity over all directions, we obtain the mean intensity $J$.

$$J = \frac{1}{4\pi} \int_{4\pi} I d\Omega \tag{3.5}$$

$$J_\nu = \frac{1}{4\pi} \int_{4\pi} I_\nu d\Omega \tag{3.6}$$

$$\tag{3.7}$$

The *specific energy density* $U_\nu$ of the radiation field can be found by dividing the energy per Hertz per square metre per second, traveling in direction $\boldsymbol{n}$, by the velocity $c$, and then integrating over all directions. This gives

$$U_\nu = \frac{4\pi J_\nu}{c}, \tag{3.8}$$

$$U = \frac{4\pi J}{c}. \tag{3.9}$$

If we weight the vector $\boldsymbol{n}$, describing the direction of propagation, by the power per unit area per steradian propagating in that direction and integrate over all solid angles, we obtain the *radiant flux*, a vector that describes the net flux of power per unit area,

$$\boldsymbol{F} = \int_{4\pi} I(\boldsymbol{n})\boldsymbol{n} d\Omega, \tag{3.10}$$

Its components $(F_x, F_y, F_z)$ represent the power per square meter flowing in the $x$, $y$, and $z$ directions respectively.

Similarly, one defines the *specific flux* by

$$\boldsymbol{F}_\nu = \int_{4\pi} I_\nu(\boldsymbol{n})\boldsymbol{n} d\Omega, \tag{3.11}$$

These quantities also have other names. Specific intensity as sometimes called *brightness*, intensity is also called *radiance* and flux is also called *irradiance*. Radio astronomers often use the symbol $B$ or $B_\nu$ for brightness and $S$ or $S_\nu$ for flux density. You may sometimes see the symbol $I$ being used for irradiance. Do not confuse this with intensity!

If we integrate the flux over the entire surface $S$ of an emitting body, we obtain the *luminosity* $L$ , the energy emitted per unit time, or in the case of specific flux, the *specific luminosity* $L_\nu$,

$$L = \int_S \boldsymbol{F} \cdot d\boldsymbol{S}. \tag{3.12}$$

The flux at the surface of a spherical object of radius $R$ emitting isotropically is therefore

$$\boldsymbol{F} = \frac{L}{4\pi R^2}\boldsymbol{r}. \tag{3.13}$$

and for any distance $r > R$,

$$\boldsymbol{F} = \frac{L}{4\pi r^2}\boldsymbol{r}. \tag{3.14}$$

## 3.1 Magnitudes

Optical astronomers often use *magnitudes* to describe the relative brightnesses of celestial objects. A magnitude $m_a$ is defined by

$$m_a = -2.5 \log_{10} F_a + C_a \tag{3.15}$$

Here $F_a$ is the flux received, at the Earth, in some wavelength or frequency band '$a$', defined by a transmission function $W_a$

$$F_a = \int_0^\infty F_\lambda W_a(\lambda) d\lambda, \tag{3.16}$$

and $C_a$ is a constant. The constant is chosen so that a particular star (Vega) has magnitude equal to zero in all wavelength bands. In the infrared part of the spectrum, the bands are chosen to avoid wavelengths of high atmospheric absorption, as shown in Fig 3.1. An example, the Johnson-Morgan photometric system described in Table 3.1. The magnitudes in these bands are often named by the band (e.g. $U \equiv m_U$, $B \equiv m_B$, etc).

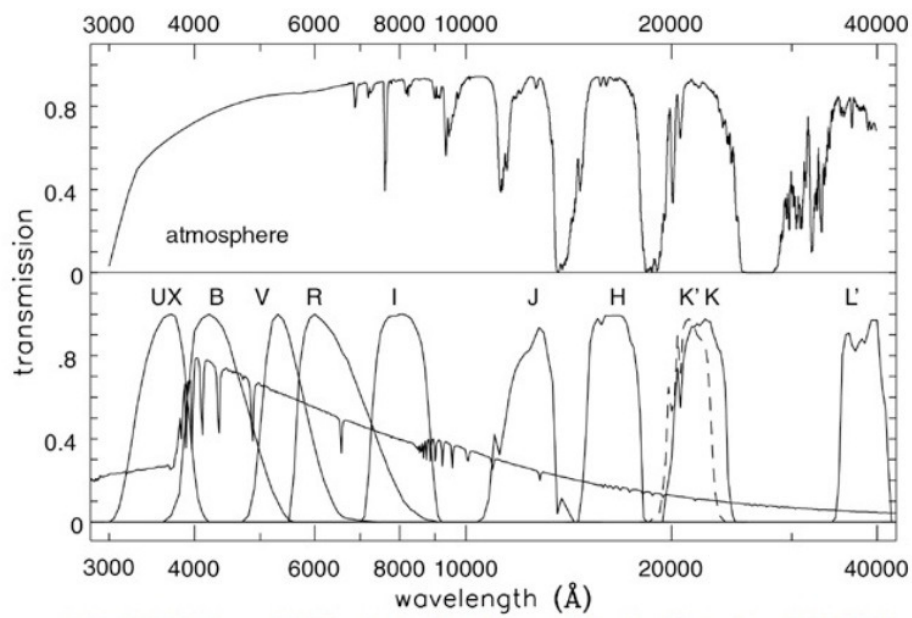



Figure 3.1: The upper panel shows atmospheric transmission. The lower panel shows transmission curves for common photometric bands. (Sparke and Gallagher).

It is also convenient to define a magnitude that is a continuous function of frequency or wavelength, which can be used to describe the spectrum or spectral energy distribution of an object. The *AB* magnitude is defined by

$$m_{\rm AB}(\nu) = -2.5 \log_{10} \left( \frac{F_\nu}{3631 \rm Jy} \right) \tag{3.17}$$

The reference flux of 3631 Jy is chosen to make the AB magnitude of Vega equals 0 at a wavelength of 555 nm.

Astronomers also use a logarithmic measure of luminosity called *absolute magnitude* $M$. This is defined as the magnitude that an object would have if it were moved to a particular reference

Table 3.1: Some common photometric bands

| Band | $\lambda_{\rm eff}$ (um) | $\Delta\lambda$ (um) | $F_\nu(m=0)$ (Jy) | $m - m_{\rm AB}(\lambda_{\rm eff})$ |
|---|---|---|---|---|
| U | 0.36 | 0.068 | 1810 | 0.756 |
| B | 0.44 | 0.098 | 4260 | -0.174 |
| g | 0.52 | 0.073 | 3730 | -0.029 |
| V | 0.55 | 0.089 | 3640 | 0.003 |
| R | 0.70 | 0.149 | 3080 | 0.179 |
| r | 0.67 | 0.106 | 4490 | -0.231 |
| I | 0.79 | 0.125 | 2550 | -0.384 |
| i | 0.79 | 0.125 | 4760 | -0.294 |
| z | 0.91 | 0.118 | 4810 | -0.305 |
| J | 1.26 | 0.38 | 1600 | 0.890 |
| H | 1.60 | 0.48 | 1080 | 1.316 |
| K | 2.22 | 0.70 | 670 | 1.885 |
| L | 3.40 | 1.20 | 312 | 2.665 |
| M | 5.00 | 5.70 | 183 | 3.244 |

distance. For stellar and extragalactic astronomy, the reference distance is 10 pc. For solar system studies, 1 AU is used. Since the distance dependence has been removed, it follows that absolute magnitude is related to the luminosity of the object,

$$M_a = -2.5\log_{10} L_a + constant. \tag{3.18}$$

where the constant depends on the wavelength band and reference distance.

Finally, there is also a logarithmic measure of intensity, called *surface brightness* $\mu_a$. It is defined as the magnitude corresponding to the flux received from one square arcsec of solid angle centred at direction $-\boldsymbol{n}$. Thus

$$\mu_a = -2.5\log_{10} I_a + C_a + 26.5721, \tag{3.19}$$

where

$$I_a = \int_0^\infty I_\lambda W_a(\lambda) d\lambda, \tag{3.20}$$

and the numerical constant is more precisely $5\log_{10}(180 \times 3600/\pi)$.

## 3.2 Photometric precision

The precision of photometric measurements is of course limited by noise. At optical and infrared wavelengths, the quantum nature of light is evident and the dominant source of noise is usually photon statistics. To first order, the arrival times of photons are uncorrelated. The number of photons received in time $\Delta t$ is a random variable having a Poisson frequency distribution. To see this, suppose that the average arrival rate of photons is $R$ (photons per second). What is the probability that no photons will arrive in time $t$? To find this, we divide the interval $t$ into a large number $N$ of equal intervals, called bins, of size $\Delta t = t/N$. The probability that no photons arrive in time $t$ is equal to the product of the probability that no photon arrives in the first bin, multiplied by the probability that no photon arrives in the second bin, etc. In the limit as $N \to \infty$, each of these probabilities is $1 - R\Delta t$. Therefore,

$$P_0(t) = \prod_{k=0}^{\infty} \left(1 - \frac{Rt}{N}\right)^N = e^{-Rt}. \tag{3.21}$$

Expressing this in terms of the expected number $x = Rt$,

$$P_0(x) = \prod_{k=0}^{\infty} \left(1 - \frac{Rt}{N}\right)^N = e^{-x}, \tag{3.22}$$

which is also called the *void probability* by cosmologists. Continuing, the probability that exactly $n$ photons arrive in time $t$ is the product of the probabilities that no photons arrive in $N - n$ bins and that one photon arrives in each of $n$ bins. Each of these photons could have arrived in any of the $N$ bins, so the arrival probability of each is $NR\Delta t = x$, However, we do not know, or care, which photon was in which bin, so we must divide the result by the number of permutations $n!$, of hte $n$ photons, in order to prevent over-counting. Taking the limit $N \to \infty$ we get the Poisson distribution

$$P_n(x) = \frac{x^n}{n!} e^{-x}. \tag{3.23}$$

Clearly

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} e^{-x} = 1, \tag{3.24}$$

as expected (*some* number of photons arrived, even if it is zero).

The mean number of photons that arrive in time $t$ is

$$\begin{aligned} \langle n \rangle &= \sum_{n=0}^{\infty} n P_n(x), \\ &= \sum_{n=1}^{\infty} x P_{n-1}(x), \\ &= x, \end{aligned} \tag{3.25}$$

And the variance is

$$\begin{aligned} \mathrm{Var(n)} = \left\langle n^2 \right\rangle - \langle n \rangle^2 &= \sum_{n=0}^{\infty} n^2 P_n(x) - x^2, \\ &= \sum_{n=0}^{\infty} \left[ n(n-1) P_n(x) + n P_n(x) \right] - x^2, \\ &= \sum_{n=2}^{\infty} x^2 P_{n-2}(x) + x - x^2, \\ &= x, \end{aligned} \tag{3.26}$$

We see that the variance equals the mean, so the best estimate of the RMS uncertainty in the photon count is the square root of the actual number of photons received. For example, if 100 photons were detected when measuring the flux from a star, the relative error in the flux would be $\sqrt{100}/100 = 0.1$. So the estimated uncertainty would be 10%, which corresponds to about 0.1 magnitudes. Thus the *signal-to-noise ratio*, the reciprocal of the relative error, is the square root of the number of detected photons.

### 3.3 Astrometric measurements

Astrometry is the measurement of precise positions of celestial objects. For example one could take an image of a field of stars and then measure the positions of each star by examining the number of photons received in each pixel of the image. Normally a star image will cover several pixels. A simple measure of the position is the *centroid* of the image, defined as

$$\langle x \rangle = \frac{1}{N} \sum_k n_k x_k,$$

$$\langle y \rangle = \frac{1}{N} \sum_k n_k y_k, \tag{3.27}$$

$$\tag{3.28}$$

where

$$N = \sum_k n_k \tag{3.29}$$

is the total number of photons. We are often interested in the accuracy of such measurements. Formally, we can compute the variance of $\langle x \rangle$, using the rules for adding errors. If $x, y, \cdots$ are uncorrelated random variables, and $f$ is some function of them, then

$$\mathrm{Var}[f(x, y, \cdots)] = \left| \frac{\partial f}{\partial x} \right|^2 \mathrm{Var}(x) + \left| \frac{\partial f}{\partial y} \right|^2 \mathrm{Var}(y) + \cdots . \tag{3.30}$$

This gives

$$\begin{aligned} \mathrm{Var}(\langle x \rangle) &= \sum_k \left( \frac{x_k}{N} \right)^2 \mathrm{Var}(n_k) - \left[ \frac{1}{N^2} \sum_k x_k n_k \right]^2 \mathrm{Var}(N), \\ &= \frac{1}{N^2} \sum_k x_k^2 n_k - \frac{1}{N} \left[ \frac{1}{N} \sum_k x_k n_k \right]^2, \\ &= \frac{1}{N} \left( \langle x^2 \rangle - \langle x \rangle^2 \right) \equiv \frac{\sigma_x^2}{N}. \end{aligned} \tag{3.31}$$

The quantity in parenthesis is the second moment of the image intensity distribution, which is the square of the characteristic size of the image, $\sigma_x$. The factor of $1/N$ is the reciprocal of the square of the signal to noise ratio (which as we have just seen is the square root of the number of photons). From this we see that the uncertainty in the $x$ position of the image centroid is equal to the characteristic width of the image divided by the signal-to-noise ratio. In the same manner we get a similar result for the $y$ direction.

# 4 Relativistic kinematics

In astrophysics, we are often dealing with relativistic particles that are being accelerated by electric or magnetic forces. This produces radiation, typically in the form of synchrotron or inverse-Compton radiation. Before examining this, we begin with a short review of Special Relativity and the concepts of spacetime and relativistic covariance.

## 4.1 Special Relativity and four-dimensional notation

Most relativistic equations are greatly simplified by the use of four-dimensional notation. An event space-time can be represented by four coordinates $\vec{x} = (x^0, x^1, x^2, x^3) \equiv (t, x, y, z) = (t, \boldsymbol{r})$, where we have set $c = 1$.

The postulates of Special Relativity are that 1) The laws of physics are the same in all inertial reference frames (i.e. moving with constant velocity) and 2) the speed of light $c$ is the same in all inertial frames. Now, imagine two frames, $O$ and $O'$, moving with respect to each other with a constant velocity. Let the the origins of the two frames coincide at $t = t' = 0$. Suppose that at exactly this time, a flash of light is emitted at the origin. According to the postulates of Special Relativity, observers in both frames see a sphere of light expanding from the origin at speed $c$. Therefore,

$$t^2 - r^2 = t'^2 - r'^2 = 0. \tag{4.1}$$

Any transformation $\vec{x'} = \Lambda\vec{x}$, relating the two frames, that satisfies (4.1) is a *Lorentz transformation*. We shall have more to say about these shortly.

Imagine two points $a$ and $b$ (called *events*) in space-time. The quantity

$$\Delta\tau = \left[(t_a - t_b)^2 - |\boldsymbol{r}_a - \boldsymbol{r}_b|^2\right]^{1/2} \tag{4.2}$$

is called the *proper time* interval between the two events. It has the same value in all inertial frames and is therefore called a *Lorentz invariant* or *scalar*. If $\Delta\tau^2 > 0$ the points are said to be separated by a *timelike* interval, if $\Delta\tau^2 < 0$ the interval is said to be *spacelike* and if $\Delta\tau^2 = 0$ it is *null*.

If the interval is time-like, a frame exists for which the two events have the same position, $x_a = x_b$, $y_a = y_b$ and $z_a = z_b$. In this frame, $\Delta\tau^2 = (t_a - t_b)^2$. This shows that *the proper time interval between two events is the time interval measured by a clock that is present at both events.*

In any other frame, moving at speed $v$ with respect to this *proper frame*, the proper time is

$$\begin{aligned}\Delta\tau &= \Delta t(1 - v^2)^{1/2} \\ &= \frac{1}{\gamma}\Delta t \end{aligned} \tag{4.3}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \tag{4.4}$$

is called the *Lorentz factor*. From this we see that the proper time interval $\Delta\tau$ is never greater than the coordinate time interval $\Delta t$. The observer in this frame sees the "proper" clock moving at speed $v$ and concludes that *moving clocks run slower*. Taking the limit as $\Delta t \to 0$, we see that

$$\frac{dt}{d\tau} = \gamma. \tag{4.5}$$

For two points separated by an infinitesimal distance and time, the proper time interval is

$$d\tau^2 = dt^2 - |d\boldsymbol{r}|^2. \tag{4.6}$$

This can be written as an inner (dot) product of two four vectors,

$$d\tau^2 = \vec{dx} \cdot \vec{dx} \equiv \sum_{jk} \eta_{jk} dx^j dx^k, \tag{4.7}$$

where

$$\eta_{jk} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.8}$$

is the *Minkowski metric.*

It is common to omit the summation symbol seen in (4.7), with the understanding that any index that appears twice in a term, once as a subscript and once as a superscript, will be summed over. This is the *Einstein summation convention.*

Under a Lorentz transformation $\Lambda$, the components of a contravariant four-vector $\vec{v}$ transform according to

$$v'^j = \Lambda^j{}_k v^k, \tag{4.9}$$

where $\Lambda^j{}_k$ is a $4 \times 4$ matrix.

There is another type of vector called a *covariant* vector that has a different transformation law

$$w'_j = w_k (\Lambda^{-1})^k{}_j. \tag{4.10}$$

An example is $\vec{\partial} = (\partial_t, \nabla)$.

From (4.9) and (4.10) we see that the *inner product* (dot product) of a covariant and a contravariant vector is a scalar,

$$\vec{w'} \cdot \vec{v'} = w'_k v'^k = w_k v^k = \vec{w} \cdot \vec{v}. \tag{4.11}$$

Writing this out we have

$$\vec{w} \cdot \vec{v} = w_0 v^0 + w_1 v^1 + w_2 v^2 + w_3 v^3. \tag{4.12}$$

Note that the signs are all positive.

The metric tensor allows us to convert covariant vectors to contravariant, and vice verca. For example,

$$\partial^j = \eta^{jk} \partial_k \tag{4.13}$$

transforms like a contravariant vector. $\eta^{jk}$ is the inverse matrix of $\eta_{jk}$, and in fact has exactly the same components. One can verify directly that the matrix product is the *Kronecker delta* $\delta^i_j$ which is defined to equal 1 if $i = j$ and 0 otherwise.

$$\eta^{ij}\eta_{jk} = \delta^j_k. \tag{4.14}$$

We will often represent a four-vector using *3+1 notation*. Instead of writing $\vec{a} = (a^0, a^1, a^2, a^3)$ we just write $\vec{a} = (a^0, \boldsymbol{a})$. The rule for multiplying two contravariant four-vectors (4.7) becomes

$$\vec{a} \cdot \vec{b} = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}. \tag{4.15}$$

Note the minus sign! Similarly, the product of two covariant vectors also has a minus sign,

$$\partial^2 = \vec{\partial} \cdot \vec{\partial} = \partial_t^2 - \nabla^2. \tag{4.16}$$

Since a Lorentz transformation must keep $d\tau^2$ invariant, it follows from (4.7) that

$$d\tau^2 = \vec{dx}' \cdot \vec{dx}' = \eta_{jk}\Lambda^j{}_l\Lambda^k{}_m dx^l dx^m, \tag{4.17}$$

Comparing this with (4.7), which, with a change of labels, can be written as

$$d\tau^2 = \eta_{lm} dx^l dx^m, \tag{4.18}$$

we see that

$$\left(\eta_{jk}\Lambda^j{}_l\Lambda^k{}_m - \eta_{lm}\right) dx^l dx^m = 0. \tag{4.19}$$

This must be true for any choice of $\vec{dx}$, which can only happen if

$$\eta_{jk}\Lambda^j{}_l\Lambda^k{}_m = \eta_{lm}. \tag{4.20}$$

This is a matrix equation, which can also be written as $\Lambda^T\eta\Lambda = \eta$. Taking the determinant of both sides and recalling that $\det(ABC\ldots) = \det(A)\det(B)\det(C)\ldots$, gives the condition

$$(\det\Lambda)^2 = 1. \tag{4.21}$$

This shows that a Lorentz transformation must have determinant of $\pm 1$. Normally one considers only *proper* Lorentz transformations, for which $\det\Lambda = 1$ (a determinant of $-1$ corresponds to a mirror reflection, or change of parity). We can also restrict ourselves to *isochronous* transformations, which have $\Lambda^0{}_0 > 0$ and therefore do not reverse the direction of time.

There is an additional condition that follows from (4.20) and that is that any Lorentz transformation can be written in the form $\Lambda = \exp(\eta L)$, where $L$ is a real antisymmetric matrix. In four dimensions, a general antisymmetric matrix contains 6 independent parameters. These correspond to the 6 degrees of freedom of a general Lorentz transformation (rotations in 3 dimensions, and boosts along three coordinates axes).

As an example, a boost with velocity $v$ in the $x$ direction is represented by

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.22}$$

It is worth noting that the four-dimensional volume elements $d^4x = dtdxdydz$, is invariant under Lorentz transformations. Volume elements transform in proportion to the Jacobian of the transformation, which is just the determinant of the transformation matrix. Since $\det \Lambda = 1$, four-dimensional volume elements are Lorentz invariant.

It also follows that the four-dimensional Dirac delta function $\delta^4(\vec{x}) = \delta(t)\delta(x)\delta(y)\delta(z)$ is also invariant. By definition,

$$\int \delta^4 \vec{x} d^4 x = 1 \tag{4.23}$$

in all Lorentz frames. Since the RHS is a scalar, so is the LHS.

## 4.2 Four-vectors

The simplest four-vector is the vector $\vec{s}$ that connects two events in spacetime, $a$ and $b$ say. It has components $\vec{s} = (t_b - t_a, \boldsymbol{r}_b - \boldsymbol{r}_a)$. The "length" of this vector is the proper time interval between the two events $s = \Delta\tau = (\Delta t^2 - |\Delta \boldsymbol{r}|^2)^{1/2}$.

Now consider a particle (or an observer) moving through space-time. The path of the particle (called the *world line*) can be represented as a time-like curve in spacetime. Points on the world line can be labeled by the proper time $\tau$ (the time indicated by a clock moving with the particle). The world line is completely defined by specifying the the four coordinates $x^j$ as a function of $\tau$, $x^j(\tau)$. Take any two points on the world line separated by an infinitesimal proper time $d\tau$. Since $d\tau$ is a scalar, the quantity

$$\vec{u} = \frac{d\vec{x}}{d\tau} \tag{4.24}$$

transforms like a contravariant vector under Lorentz transformations. It is called the *four velocity* of the particle. Geometrically, it is a four-dimensional vector that is tangent to the world line.

It is easy to get a formula for the four-velocity of a particle in any inertial frame. Using the chain rule and (4.5),

$$\vec{u} = \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}(t, \boldsymbol{x}) = \gamma(1, \boldsymbol{v}) \tag{4.25}$$

Note that the four-velocity has unit "length",

$$\vec{u} \cdot \vec{u} = \gamma^2(1, \boldsymbol{v}) \cdot (1, \boldsymbol{v}) = \gamma^2(1 - \boldsymbol{v} \cdot \boldsymbol{v}) = 1. \tag{4.26}$$

and that in the rest frame of the particle, it is just $(1, \mathbf{0})$.

We have been talking about the motion a particle, but we could equally-well have been talking about the motion of an observer. Many relativistic calculations are simplified by use of the four-velocity of an observer.

The rest mass $m$ of a particle is clearly a Lorentz scalar. Why? Because we have specified the frame! All observers agree that when the particle is at rest it has mass $m$. If we multiply a four-vector by a scalar, the resulting four-component object also transforms according to (4.9) and is therefore a four-vector. Multiplying the four-velocity of a particle by the particle's rest mass gives another four vector, the *four-momentum* of the particle.

$$\vec{p} = m\vec{u} \tag{4.27}$$

How should we interpret the components of the four-momentum? The timelike component,

$$p^0 = mu^0 = \gamma m = m(1 - v^2)^{-1/2} = m + \frac{1}{2}mv^2 + \cdots . \tag{4.28}$$

Putting in the missing factors of $c$ we see that this is just the total energy of the particle

$$p^0 = mc^2 + \frac{1}{2}mv^2 + \cdots = E \tag{4.29}$$

The space-like part of the four momentum is the relativistic three-momentum

$$\boldsymbol{p} = \gamma m\boldsymbol{v}. \tag{4.30}$$

Therefore, we may write $\vec{p} = (E, \boldsymbol{p})$.

Let's compute the norm of $\vec{p}$ (the square of the length),

$$\vec{p} \cdot \vec{p} = (E, \boldsymbol{p}) \cdot (E, \boldsymbol{p}) = E^2 - p^2. \tag{4.31}$$

But this must also equal

$$\vec{p} \cdot \vec{p} = m^2 \vec{u} \cdot \vec{u} = m^2. \tag{4.32}$$

Comparing the two, we obtain the relativistic energy equation,

$$E^2 = p^2c^2 + m^2c^4. \tag{4.33}$$

## 4.3 Doppler shift

As a second example, consider the problem of determining the frequency shift of light emitted by a moving source. The relevant four-vectors are the four-velocity of the source, and the wave vector describing the propagating radiation. In the rest frame of the source, $\vec{u} = \omega'(1, \boldsymbol{n}')$ and $\vec{u} = (1, \boldsymbol{0})$, where here the prime denotes the rest-frame. Therefore

$$\vec{u} \cdot \vec{k} = \omega'. \tag{4.34}$$

In a frame in which the source moves with velocity $\boldsymbol{v}$, we have $\vec{u} = \gamma(1, \boldsymbol{v})$ and $\vec{k} = \omega(1, \boldsymbol{n})$. Therefore

$$\vec{u} \cdot \vec{k} = \omega\gamma(1 - \boldsymbol{n} \cdot \boldsymbol{v}) = \omega\gamma(1 - v\cos\theta), \tag{4.35}$$

where $\theta$ is the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$. Since these are the same scalars, they must be equal. Hence

$$\omega = \frac{\omega'}{\gamma(1 - v\cos\theta)}, \tag{4.36}$$

which is the *relativistic Doppler relation*.

If the source is moving directly towards the observer, $\cos\theta = 0$ and (4.36) reduces to

$$\omega = \omega' \sqrt{\frac{1+v}{1-v}}, \tag{4.37}$$

which of course is a *blue shift*. If the source is moving directly away from the observer, replace $v \to -v$ to get

$$\omega = \omega' \sqrt{\frac{1-v}{1+v}}, \tag{4.38}$$

which is now corresponds to a *redshift*.

In the ultrarelativistic limit, $v \to 1$, (4.37) and (4.38) become

$$\omega \simeq 2\gamma\omega', \tag{4.39}$$

$$\omega \simeq \omega'/2\gamma. \tag{4.40}$$

## 4.4 Aberration

One could turn this around and say that instead, the source is at rest and the observer is moving with velocity $-\boldsymbol{v}$. In that case we would have

$$\omega' = \frac{\omega}{\gamma(1 + v\cos\theta')}. \tag{4.41}$$

Both expressions are correct. Combining them we find

$$\gamma^2(1 - v\cos\theta)(1 + v\cos\theta') = 1. \tag{4.42}$$

which gives a relation between the angle $\theta'$ seen in the rest frame of the source and the angle $\theta$ in the frame of the observer. Thus we obtain the *relativistic aberration relations*,

$$\cos\theta = \frac{\cos\theta' + v}{1 + v\cos\theta'} \tag{4.43}$$

$$\cos\theta' = \frac{\cos\theta - v}{1 - v\cos\theta}. \tag{4.44}$$

Consider a source emitting radiation isotropically in its rest frame. In this frame, half of the radiation is emitted into the hemisphere $\theta' < \pi/2$. As seen in a frame in which the source is moving, the same photons are confined to a cone of half angle $\theta$. From (4.43), $\cos\theta = v$ therefore

$$\sin\theta = \sqrt{1 - v^2} = \frac{1}{\gamma}. \tag{4.45}$$

We see that the radiation is directed forward, in the direction of motion of the source, with half of photons confined to a code of semi-angle $\sim 1/\gamma$. This, when combined with the Doppler effect and time dilation, greatly increases the flux of radiation in the forward direction, a phenomenon known as *relativisitic beaming*.

All relativistic equations can be written in terms of scalars, four-vectors and more-general objects such as tensors and spinors. Such equations remain unchanged under Lorentz transformations and are said to be *relativisticaly covariant*.

We shall have occasion to use other four-vectors, many of which are listed in Table 4.1.

Table 4.1: Some four-vectors ($c = 1$)

| Name | Definition |
|---|---|
| interval | $\vec{s} = (\Delta t, \Delta x, \Delta y, \Delta z)$ |
| four-velocity | $\vec{u} = \dfrac{d\vec{x}}{d\tau} = \gamma(1, \boldsymbol{v}) = \gamma(1, v_x, v_y, v_z)$ |
| four-acceleration | $\vec{a} = \dfrac{d\vec{u}}{d\tau}$ |
| four-momentum | $\vec{p} = m\vec{u} = \gamma(E, \boldsymbol{p})$ |
| four-force | $\vec{f} = \dfrac{d\vec{p}}{d\tau}$ |
| four-frequency | $\vec{k} = (\omega, \boldsymbol{k}) = \omega(1, \boldsymbol{n})$ |
| four-current | $\vec{j} = (\rho, \boldsymbol{J})$ |
| four-potential | $\vec{A} = (\phi, \boldsymbol{A})$ |
| four-derivative | $\vec{\partial} = \left(\dfrac{\partial}{\partial t}, \boldsymbol{\nabla}\right) = (\partial_t, \partial_x, \partial_y, \partial_z)$ |

# 5 Electrodynamics

## 5.1 Maxwell's equations in four dimensions

Electric charge $q$ is observed to be the same for all observers, and is therefore a Lorentz scalar. An extended charge is represented by the charge density $\rho$. Thus, the charge contained within a volume $dV$ is

$$dq = \rho dV. \tag{5.1}$$

Now we know that a 4-volume $d^4x = dtdV$ is also a scalar, so $\rho$ must transform in the same way as $dt$, namely as the 0-component of a four vector. To find the other three components, consider the equation of conservation of electric charge, $\partial_t \rho + \nabla \cdot \boldsymbol{J} = 0$. This can be written in four dimensions as

$$\vec{\partial} \cdot \vec{j}, \tag{5.2}$$

where

$$\vec{j} = (\rho, \boldsymbol{J}) \tag{5.3}$$

is clearly a contravariant four-vector, called the *four-current*.

According to Maxwell's equations, this current gives rise to a potential which (as we shall see) can be represented by the four-potential

$$\vec{A} = (\phi, \boldsymbol{A}). \tag{5.4}$$

The quantity

$$F^{jk} = \partial^j A^k - \partial^k A^j. \tag{5.5}$$

is an object that evidently has the transformation law

$$F'^{jk} = \Lambda^j{}_l \Lambda^k{}_m F^{lm}. \tag{5.6}$$

This is different from a vector. It is a *tensor* of rank 2.

$F^{jk}$ is called the *Maxwell tensor*. From the definition we see that it is antisymmetric ($F^{jk} = -F^{kj}$) and therefore contains 6 independent components ((16 − 4)/2. By comparing this definition with that of the scalar and vector potentials (Table 1.3), one can verify that $F^{jk}$ has components

$$F^{jk} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{5.7}$$

and that Maxwell's equations can be written as

$$\partial_j F^{jk} = j^k, \tag{5.8}$$

$$\partial^i F^{jk} + \partial^j F^{ki} + \partial^k F^{ij} = 0. \tag{5.9}$$

Usually it is simpler to work directly with the potentials. Substituting (5.5) into (5.8) we get

$$\partial^2 A^k - \partial_j \partial^k A^j = j^k. \tag{5.10}$$

to simplify this, observe from (5.5) that, because partial derivatives commute, we could add the gradient of any scalar $\chi(\vec{x})$ to $\vec{A}$ and it would not change the Maxwell tensor (and therefore the electric and magnetic fields) at all,

$$\begin{aligned}\partial^j(A^k + \partial^k \chi) - \partial^k(A^j + \partial^j \chi) &= \partial^j A^k - \partial^k A^j + (\partial^j \partial^k - \partial^k \partial^j)\chi \\ &= \partial^j A^k - \partial^k A^j\end{aligned} \tag{5.11}$$

The substitution

$$\vec{A} \rightarrow \vec{A} + \vec{\partial}\chi \tag{5.12}$$

is called a *gauge transformation*.

If we choose a function $\chi$ that satisfies $\partial^2 \chi = -\vec{\partial} \cdot \vec{A}$, then the gauge transformation will result in

$$\vec{\partial} \cdot \vec{A} = 0. \tag{5.13}$$

This is called the *Lorentz gauge*. In the Lorentz gauge, The inhomogeneous Maxwell equations, (5.8) become a four-dimensional wave equation relating the four-current to the four-potential,

$$\partial^2 A^k = j^k. \tag{5.14}$$

It is not hard to show that the homogeneous equations (5.9) are automatically satisfied because of (5.5), so (5.14) represents all of Maxwell's equations).

## 5.2 Transformation of electromagnetic fields

By expanding (5.6) one can obtain the transformation laws for the electric and magnetic fields. For the case of a boost with velocity $\boldsymbol{v}$ the result is

$$\boldsymbol{E}'_\perp = \gamma\left(\boldsymbol{E}_\perp + \boldsymbol{v} \times \boldsymbol{B}\right), \qquad \boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel \tag{5.15}$$

$$\boldsymbol{B}'_\perp = \gamma\left(\boldsymbol{B}_\perp + \boldsymbol{v} \times \boldsymbol{E}\right), \qquad \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel \tag{5.16}$$

where $\perp$ and $\parallel$ denote components perpendicular and parallel to $\boldsymbol{v}$, respectively.

## 5.3 Electromagnetic invariants

From the Maxwell tensor, we can form two scalars,

$$\frac{1}{2} F^{jk} F_{jk} = B^2 - E^2, \tag{5.17}$$

$$\frac{1}{2} \varepsilon_{jklm} F^{jk} F^{lm} = -\boldsymbol{E} \cdot \boldsymbol{B}. \tag{5.18}$$

where we have introduced the *Levi-Civita tensor* defined by

$$\varepsilon_{j_1, j_2, \cdots, j_n} = \begin{cases} 1 & \text{if} j_1, j_2, \cdots, j_n \text{ is an even permutation of } 0, 1, \cdots, n \\ -1 & \text{if} j_1, j_2, \cdots, j_n \text{ is an odd permutation of } 0, 1, \cdots, n \\ 0 & \text{otherwise} \end{cases} \tag{5.19}$$

## 5.4 The Lorentz force

Recall that the nonerlativistic motion of a charge $q$ and mass $m$ in an electromagnetic field is given by

$$m\dot{\boldsymbol{v}} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{5.20}$$

In the rest frame of the particle, this becomes $m\dot{\boldsymbol{v}} = q\boldsymbol{E}$ which matches the tensor equation

$$\frac{dp^j}{d\tau} = qF^{jk}u_k. \tag{5.21}$$

Thus, the *Lorentz four-force* acting on a charge $q$ moving with four-velocity $\vec{u}$ is

$$f^j = qF^{jk}u_k \tag{5.22}$$

## 5.5 Lienard-Wiechert potentials

Let us now try to solve Maxwell's equations (5.14) for the case of a moving point charge $q(t)$. Start with the equation for the 0 component, which is

$$\partial^2\phi = \rho \tag{5.23}$$

To simplify this, choose a reference frame in which the charge is momentarily at rest, at the origin. The charge density $\rho$ can then be written as

$$\rho = q\delta^3(\boldsymbol{r}), \tag{5.24}$$

where $\delta^3(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$. Spherical symmetry tells us that the solution must be some function of $r$ and $t$ only, so we write the Laplacian operator in spherical polar coordinates,

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right)\phi = \rho(t, r) \tag{5.25}$$

For any $r > 0$, the RHS is zero. The substitution $\phi(t, r) = f(t, r)/r$ leads to

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right) f = 0 \qquad (r > 0). \tag{5.26}$$

This is a one-dimensional wave equation that has as a solution any function of $t-r$ or $t+r$. We are only interested in waves that propagate forwards in time so we take the *retarded* solution $f(t-r)$. Therefore $\phi = f(t-r)/r$.

The form of the function $f(t-r)$ can be determined by observing that $\phi \to \infty$ as $r \to 0$. Therefore the derivative with respect to $r$ in (5.25) increases much faster than does the time derivative. In the limit, the equation becomes

$$-\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi}{\partial r} = q\delta^3(\boldsymbol{r}), \qquad r \to 0. \tag{5.27}$$

We recognize this as the Coulomb problem of electrostatics, which has the solution $\phi = q/4\pi r$. Therefore, the general solution must be

$$\phi(t,r) = \left[\frac{q}{4\pi r}\right]_{t-r}. \tag{5.28}$$

where the bracket notation tells us that the field at $(t, \boldsymbol{r})$ is determined by the position of the charge at the retarded time $t - r$.

Now, we would like to find the relativistic equation. The LHS is the 0-component of the four-vector, $\vec{A}$. So we need to form a four-vector on the RHS that has the correct limit as $v \to 0$. The only relevant four-vectors that we have are the four-velocity $\vec{u}$ of the charge and the interval $\vec{s}$ separating the charge and the observer. In the frame we chosen, in which the charge is at rest, $\vec{u} = (1, \mathbf{0})$, and $\vec{s} = (t, \boldsymbol{r})$. Since the wave propagates at the speed of light, it follows that $t = r$, and we can write $\vec{s} = (r, \boldsymbol{r})$. Therefore, the required relativistic equation is

$$\vec{A} = \left[\frac{q\vec{u}}{4\pi\vec{u}\cdot\vec{s}}\right]_{t-r}. \tag{5.29}$$

These solutions are called the *Lienard-Wiechert potentials.*

# 6 Electromagnetic waves

## 6.1 Plane waves

In free space (5.9) reduce to the vacuum Maxwell equations,

$$\partial^2 A^j = 0 \tag{6.1}$$

It can be verified by direct substitution that a solution is the plane wave

$$A^j(\vec{x}) = \mathrm{Re}\left[\mathcal{A}^j e^{i\vec{k}\cdot\vec{x}}\right], \tag{6.2}$$

where $\mathcal{A}^j$ are four complex coefficients and $\vec{k} = (\omega, \boldsymbol{k})$ is a constant vector satisfying,

$$k^2 = k^j k_j = 0. \tag{6.3}$$

Thus, $\omega^2 = \boldsymbol{k}^2$. Here the notation Re stands for the real part of a complex quantity. As long as we conduct only linear operations, we can work directly with the complex equations with the understanding that we will take the real part at the end.

The Lorentz gauge condition (5.13) tell us that

$$\vec{k} \cdot \vec{\mathcal{A}} = k^j \mathcal{A}_j = 0. \tag{6.4}$$

The Maxwell tensor for this wave can be found directly from the definition (5.5),

$$F^{jk} = \partial^j \mathcal{A}^k e^{i\vec{k}\cdot\vec{x}} - \partial^k \mathcal{A}^j e^{i\vec{k}\cdot\vec{x}} \tag{6.5}$$

$$= i(k^j \mathcal{A}^k - k^k \mathcal{A}^j) e^{i\vec{k}\cdot\vec{x}} \tag{6.6}$$

## 6.2 Electric and magnetic fields

From (5.7) we see that the electric field components are given by

$$\begin{aligned} E^\alpha &= F^{\alpha 0} = i(k^\alpha \mathcal{A}^0 - k^0 \mathcal{A}^\alpha) e^{i\vec{k}\cdot\vec{x}}, \\ &= \mathcal{E}^\alpha e^{i\vec{k}\cdot\vec{x}}, \end{aligned} \tag{6.7}$$

where $\boldsymbol{\mathcal{E}}$ is a three-dimensional vector having components $\mathcal{E}^\alpha = k^\alpha \mathcal{A}^0 - k^0 \mathcal{A}^\alpha$. Similarly, the magnetic field is

$$\begin{aligned} B^\alpha &= -\frac{1}{2}\varepsilon^{\alpha\beta\gamma} F_{\beta\gamma}, \\ &= -\frac{i}{2}\varepsilon^{\alpha\beta\gamma}(k_\beta \mathcal{A}_\gamma - k_\gamma \mathcal{A}_\beta) e^{i\vec{k}\cdot\vec{x}}, \\ &= \mathcal{B}^\alpha e^{i\vec{k}\cdot\vec{x}}, \end{aligned} \tag{6.8}$$

It is easy to verify that

$$\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}} = k_\alpha \mathcal{E}^\alpha = 0, \tag{6.9}$$
$$\boldsymbol{k} \cdot \boldsymbol{\mathcal{B}} = k_\alpha \mathcal{B}^\alpha = 0, \tag{6.10}$$

which shows that the electric and magnetic fields are perpendicular to the direction of propagation.

Direct substitution shows that the field invariants for the wave are both zero,

$$B^2 - E^2 = F^{jk} F_{jk} = 0, \tag{6.11}$$
$$-\boldsymbol{E} \cdot \boldsymbol{B} = \frac{1}{2} \varepsilon_{jklm} F^{jk} F^{lm} = 0, \tag{6.12}$$

so we see that the electric and magnetic field vectors are orthogonal and have equal amplitude.

## 6.3 Energy and momentum

The energy and momentum of the electromagnetic field is described by the *energy-momentum tensor*,

$$T^{jk} = -F^{jl} F_l{}^k + \frac{1}{4} \eta^{jk} F^{lm} F_{lm} \tag{6.13}$$

(See for example Landou and Lifschitz, *The Classical Theory of Fields*). The $T^{00}$ component of this tensor is the energy density $U$, and the components $T^{0\alpha}, \alpha = 1, 2, 3$ are the components of the momentum flux vector. Since photons are massless, it follows from (4.33) that $E^2 = \boldsymbol{p}^2 c^2$, so these components also the energy flux vector $\boldsymbol{F}$, Eqn. (3.10).

The energy-momentum tensor is quadratic in the fields, so we must take the real part before multiplying. The fields oscillate with frequency $\omega$, so to determine the energy, we average over the period $2\pi/\omega$. If $A$ and $B$ are two complex quantities that vary sinusoidally, one can show that the time average

$$\langle \text{Re } A \text{ Re } B \rangle = \frac{1}{2} \text{Re}(AB^*). \tag{6.14}$$

Substituting from (6.6) and using (6.14), we find,

$$\begin{aligned} T^{jk} &= -\frac{1}{2} F^{jl} F^*{}_l{}^k \\ &= \frac{1}{2} (k^j \mathcal{A}^l - k^l \mathcal{A}^j)(k_l \mathcal{A}^{*k} - k^k \mathcal{A}^*_l), \\ &= \frac{1}{2} |\mathcal{A}|^2 k^j k^k. \end{aligned} \tag{6.15}$$

The energy density is therefore

$$U = T^{00} = \frac{1}{2} |\mathcal{A}|^2 k^0 k^0 = \frac{1}{2} |\mathcal{A}|^2 \omega^2, \tag{6.16}$$

and the flux is

$$F^\alpha = T^{0\alpha} = \frac{1}{2}|\mathcal{A}|^2\omega k^\alpha = Un^\alpha, \tag{6.17}$$

where $\boldsymbol{n} = \boldsymbol{k}/|\boldsymbol{k}| = \boldsymbol{k}/\omega$ is a unit vector pointing in the direction of propagation.

We could also have computed these results from the classical formulae for energy density and the Pointing vector (1.3). For example,

$$\begin{aligned} U &= \frac{1}{2}(E^2 + B^2), && (6.18)\\ &= E^2, && (6.19)\\ &= \frac{1}{2}(k^\alpha \mathcal{A}^0 - k^0\mathcal{A}^\alpha)(k_\alpha \mathcal{A}^{*0} - k^0\mathcal{A}^*_\alpha), \\ &= \frac{1}{2}(\omega^2|\mathcal{A}^0|^2 - 2\omega\mathcal{A}^0\boldsymbol{k}\cdot\boldsymbol{\mathcal{A}}^* + \omega^2\boldsymbol{\mathcal{A}}\cdot\boldsymbol{\mathcal{A}}^*), \\ &= \frac{1}{2}|\mathcal{A}|^2\omega^2. && (6.20)\end{aligned}$$

## 6.4 Polarization and coherence

We have seen that in a plane wave, the electric and magnetic fields are perpendicular to the direction of propagation, which means that they are *transverse* waves. The direction of the electric field can be represented by a time-independent unit three-vector $\boldsymbol{\varepsilon}$. If this vector is vector is real the direction of the electric vector does not change (it oscillates between positive and negative values) and we say that the radiation is linearly polarized. If it is complex, the direction of the electric vector rotates with time at frequency $\omega$ and we have elliptical or circular polarization.

In general, the radiation may consist of many independent photons, which have no well-defined phase relationship to each other. Such radiation is said to be incoherent. (In quantum mechanical terms we say that the radiation is in a "mixed state"). For either coherent or incoherent radiation, we can construct a two dimensional matrix, defined in the transverse plane,

$$\mathcal{I}^{\alpha\beta} = \left\langle \mathcal{E}^\alpha \mathcal{E}^{*\beta} \right\rangle \tag{6.21}$$

with trace $\mathcal{I} = \mathcal{I}^\alpha{}_\alpha$, and a dimensionless matrix called the *polarization tensor*

$$\rho^{\alpha\beta} = \frac{\mathcal{I}^{\alpha\beta}}{\mathcal{I}} \tag{6.22}$$

The polarization tensor is *hermitian*, $\rho^{\alpha\beta} = \rho^{*\beta\alpha}$. Therefore, it can characterized by three independent real parameters $\xi_1, \xi_2, \xi_3$ which describe the degree and orientation of linear and circular polarization.

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\xi_1 & \xi_2 - i\xi_3 \\ \xi_2 + i\xi_3 & 1-\xi_1 \end{pmatrix}. \tag{6.23}$$

Each of these parameters range from $-1$ to 1, although the sum of their squares cannot exceed unity. (To see this observe that $\det\mathcal{I} \geqslant 0$ and $\det\rho = 1 - \xi_i^2 - \xi_2^2 - \xi_3^2$. To prove the first statement, rotate the $x, y$ coordinates until $\mathcal{E}^1 = 0$.) They are related to the *Stokes parameters* $(I, Q, U, V)$ of classical optics via the relations

$$\begin{aligned} I &= \mathcal{I} \\ Q &= \mathcal{I}\xi_1, \\ U &= \mathcal{I}\xi_2, \\ V &= \mathcal{I}\xi_3, \end{aligned} \tag{6.24}$$

The tensor $\mathcal{I}^{\alpha\beta}$, and the Stokes parameters, are additive for superpositions of incoherent radiation. The *degree of polarization* is given by

$$\begin{aligned} P &= \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \\ &= \frac{1}{I}\sqrt{Q^2 + U^2 + V^2}. \end{aligned} \tag{6.25}$$

# 7 Radiation by a moving charge

## 7.1 Larmor's formula

We are now ready to analyze the radiation emitted by a moving charge. The Lienard-Wiechert potentials (5.29) give the four-potential at any point $\vec{x}$ in spacetime. To get Maxwell tensor, we must differentiate this field with respect to $x^j$. While the four-velocity of the particle is not an explicit function of $x^j$, there is an implicit dependence because the velocity must be evaluated at the retarded time, which depends on $x^j$. To find this dependence, let the four-position of the particle be $\vec{y}(\tau)$ and differentiate $s^2 = (\vec{x} - \vec{y})^2$,

$$\begin{aligned} \partial_j s^2 &= 2s_k(\delta_j^k - u^k \partial_j \tau) \\ &= 2(s_j - s_k u^k \partial_j \tau). \end{aligned} \tag{7.1}$$

Because $s^2 = 0$ for all solutions of the wave equation, the RHS must be zero, thus

$$\partial_j \tau = \frac{s_j}{s_k u^k}. \tag{7.2}$$

With this result we can now evaluate derivatives. For example,

$$\partial_j u^k = a^k \partial_j \tau = \frac{a^k s_j}{s_l u^l}, \tag{7.3}$$

$$\begin{aligned} \partial_j s^k &= \partial_j (x^k - y^k), \\ &= \delta_j^k - u^k \frac{s_j}{s_l u^l}, \end{aligned} \tag{7.4}$$

where $a^k = du^k/d\tau$ is the four-acceleration. Using these relations, we obtain

$$F^{jk} = \frac{q}{4\pi}\left[\frac{s^j a^k - s^k a^j}{(s_l u^l)^2} - \frac{s_l a^l - 1}{(s_l u^l)^3}(s^j u^k - s^k u^j)\right]. \tag{7.5}$$

Since $\vec{s} = r(1, \boldsymbol{n})$, we see that all terms falls off as $r^{-1}$ except the term involving $-1$, which falls off as $r^-2$. Far from the source, we can neglect that term, giving for the radiation field

$$F^{jk} = \frac{q}{4\pi}\left[\frac{s^j a^k - s^k a^j}{(s_l u^l)^2} - \frac{s_l a^l}{(s_l u^l)^3}(s^j u^k - s^k u^j)\right]. \tag{7.6}$$

The energy momentum tensor can now be found,

$$T^{jk} = -\frac{q^2}{16\pi^2}\frac{(\vec{s}\cdot\vec{u})^2 a^2 + (\vec{s}\cdot\vec{a})^2}{(\vec{s}\cdot\vec{u})^6} s^k s^k, \tag{7.7}$$

In order to interpret these results in terms of three-dimensional quantities, the following relations are useful,

$$\vec{a} = \gamma^2[\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v}, \gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{v} + \boldsymbol{a}], \tag{7.8}$$

$$a^2 = -\gamma^6(\boldsymbol{a}\cdot\boldsymbol{v})^2 - \gamma^4\boldsymbol{a}^2, \tag{7.9}$$

$$\vec{s}\cdot\vec{u} = r\gamma(1 - \boldsymbol{n}\cdot\boldsymbol{v}), \tag{7.10}$$

$$\vec{s}\cdot\vec{a} = r\gamma^4(\boldsymbol{a}\cdot\boldsymbol{v}(1 - \boldsymbol{n}\cdot\vec{v}) - r\gamma^2(\boldsymbol{n}\cdot\boldsymbol{a}). \tag{7.11}$$

Using these, we find

$$T^{jk} = \frac{q^2}{16\pi^2} \frac{\gamma^2(1-\boldsymbol{n}\cdot\boldsymbol{v})^2\boldsymbol{a}^2 + 2\gamma^2(\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{a}\cdot\boldsymbol{v})(1-\boldsymbol{n}\cdot\boldsymbol{v}) - (\boldsymbol{n}\cdot\boldsymbol{a})^2}{r^4\gamma^2(1-\boldsymbol{n}\cdot\boldsymbol{v})^6} s^k s^k, \tag{7.12}$$

In the (instantaneous) rest frame of the source, this simplifies to

$$T^{jk} = \frac{q^2}{16\pi^2} \frac{\boldsymbol{a}^2 - (\boldsymbol{n}\cdot\boldsymbol{a})^2}{r^4} s^k s^k, \tag{7.13}$$

$$= \frac{q^2}{16\pi^2} \frac{\boldsymbol{a}^2 \sin^2\varphi}{r^4} s^k s^k, \tag{7.14}$$

where $\varphi$ is the angle between the (three-dimensional) acceleration and the direction to the observer. In this frame, the energy density and flux of the radiation are

$$U = T^{00} = \frac{q^2}{16\pi^2} \frac{\boldsymbol{a}^2 \sin^2\varphi}{r^2}, \tag{7.15}$$

$$F = U\boldsymbol{n}. \tag{7.16}$$

The power radiated per unit solid angle, in direction $\varphi$, is therefore

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2} \boldsymbol{a}^2 \sin^2\varphi. \tag{7.17}$$

Integrating this over solid angle gives the total power radiated by the source,

$$\begin{aligned} P &= \frac{q^2\boldsymbol{a}^2}{16\pi^2} \int_{4\pi} \sin^2\varphi d\Omega, \\ &= \frac{q^2\boldsymbol{a}^2}{8\pi r^2} \int_{-1}^{1} (1-x^2) dx, \\ &= \frac{q^2\boldsymbol{a}^2}{6\pi}. \end{aligned} \tag{7.18}$$

These results were first obtained by J. J. Larmor in 1897, using a non-relativistic analysis.

## 7.2 Relativistic Larmor formula

To find the relativistic equivalent, we use the fact that the emitted power $P$ is Lorentz invariant for any emitter that emits with front-back symmetry in its instantaneous rest frame. For such an emitter, the momentum emitted in time $dt$ is zero, therefore under a Lorentz transformation, the energy $dE$ is proportional to $\gamma$, as is $dt$. Therefore, $P = dE/dt$ is invariant. The RHS can be written in a covariant form by noting from (7.8) that in the rest frame, $\vec{a} = (0, \boldsymbol{a})$. Therefore, the relativistic Larmor formula is

$$P = -\frac{q^2 a^2}{6\pi}. \tag{7.19}$$

This can be written in terms of the three-acceleration using (7.9)

$$\begin{aligned} P &= \frac{q^2\gamma^4}{6\pi}[\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2 + \boldsymbol{a}^2] \\ &= \frac{q^2\gamma^4}{6\pi}(a_\perp^2 + a_\parallel^2 + \gamma^2 v^2 a_\parallel^2) \\ &= \frac{q^2\gamma^4}{6\pi}(a_\perp^2 + \gamma^2 a_\parallel^2) \end{aligned} \tag{7.20}$$

where $a_\perp$ and $a_\parallel$ are the components of $\boldsymbol{a}$ perpendicular and parallel to $\boldsymbol{v}$, respectively.

The angular distribution of this power can be found from (7.12),

$$\begin{aligned} \frac{dP}{d\Omega} &= r^2 T^{00}, \\ &= \frac{q^2}{16\pi^2}\frac{\gamma^2(1-\boldsymbol{n}\cdot\boldsymbol{v})^2\boldsymbol{a}^2 + 2\gamma^2(\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{a}\cdot\boldsymbol{v})(1-\boldsymbol{n}\cdot\boldsymbol{v}) - (\boldsymbol{n}\cdot\boldsymbol{a})^2}{\gamma^2(1-\boldsymbol{n}\cdot\boldsymbol{v})^6} \end{aligned} \tag{7.21}$$

Two special cases are of particular interest. If the acceleration is parallel to the velocity, we have

$$\frac{dP}{d\Omega} = \frac{q^2 a_\parallel^2 \sin^2\theta}{16\pi^2(1-v\cos\theta)^6} \tag{7.22}$$

where $\theta$ is the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$. And, if the acceleration is perpendicular to the velocity we have

$$\frac{dP}{d\Omega} = \frac{q^2 a_\perp^2}{16\pi^2(1-v\cos\theta)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\cos\theta)^2}\right], \tag{7.23}$$

where $\phi$ is the angle between $\boldsymbol{a}$ and the projection of $\boldsymbol{n}$ on the plane perpendicular to $\boldsymbol{v}$.

For highly-relativistic particles, $v \simeq 1$ and we have $1 - v\cos\theta \simeq (1+\gamma^2\theta^2)/2\gamma^2$. The equations become

$$\frac{dP}{d\Omega} \simeq \frac{4q^2 a_\parallel^2\gamma^{12}\theta^2}{\pi^2(1+\gamma^2\theta^2)^6} \tag{7.24}$$

$$\frac{dP}{d\Omega} \simeq \frac{q^2 a_\perp^2\gamma^8(1-2\gamma^2\theta^2\cos 2\phi + \gamma^4\theta^4)}{4\pi^2(1+\gamma^2\theta^2)^6}, \tag{7.25}$$

## 7.3 Relativistic invariants

We have seen one example of an invariant: The radiated power is invariant under Lorentz transformations, provided that the emitted radiation has front-back symmetry in the emitter's rest frame (and therefore zero net momentum component in the direction of the velocity). Consider now a group of $N$ particles (which could be photons) that at any given time have a small spread in position and momentum. In the centre-of-mass frame, the particles occupy a volume $d^3x' = dx'^1dx'^2dx'^3$ and momentum space volume $d^3p' = dp'^1dp'^2dp'^3$. In a moving frame, the volume appears smaller, $d^3x = d^3x'/\gamma$. The momentum transforms like a four-vector, so under a boost in the $x$ direction, $dp^1 = \gamma(dp'^1 + dE')$. However, $E' \simeq mc^2 + \boldsymbol{p}'^2/(2mc^2)$ so $dE' = (2p'^1/mc^2)dp'^1 \ll dp'^1$. Therefore,

$d^3p = \gamma d^3p'$. From this we see that the phase space volume $d^3xd^3p$ is invariant, and therefore the phase space density

$$f = \frac{dN}{d^3xd^3p} \tag{7.26}$$

is also invariant.

It is not hard to show that the phase space density of photons is related to the specific intensity. Consider a group of $dN$ photons propagating in the $z$ direction with just a small spread of directions and frequencies. The energy carried by the photons is

$$\begin{aligned} dE &= \hbar\omega dN = I_\nu dA dt d\Omega d\nu \\ &= \frac{I_\nu}{2\pi c\omega^2} dA dz \boldsymbol{k}^2 dk d\Omega \\ &= \frac{I_\nu}{c\omega^2} d^3xd^3p \end{aligned} \tag{7.27}$$

Therefore,

$$f = \frac{I_\nu}{c\hbar\omega^3}, \tag{7.28}$$

so $I_\nu/\nu^3$ is invariant.

The *emission coefficient* $j_\nu(\boldsymbol{n})$ is defined as the power emitted per cubic metre, per Hz, per steradian, in direction $\boldsymbol{n}$. Thus,

$$\begin{aligned} j_\nu &= \frac{dP}{d^3xd\Omega d\nu} \\ &= \frac{2\pi\omega^2 dP}{d^3x\boldsymbol{k}^2 dk d\Omega} \\ &= \frac{2\pi\omega^2 dP}{d^3xd^3p}. \end{aligned} \tag{7.29}$$

If the emitter has front-back symmetry, $dP$ is invariant, and therefore $j_\nu/\nu^2$ is invariant.

# 8 Thompson and Compton scattering

An electromagnetic wave impinging on a charged particle, such as an electron, creates an oscillating motion of the charge. In turn, the oscillating charge generates radiation. This process is known as scattering. If the motion of the charge is nonrelativistic, the process is called Thompson scattering. The relativistic case is called Compton scattering.

## 8.1 Thompson scattering

Consider a linearly-polarized monochromatic plane wave incident on a particle of charge $q$ and mass $m$ initially at rest. The electric field at the particle has the form

$$\boldsymbol{E} = \mathrm{Re}[\boldsymbol{\mathcal{E}} e^{i\omega t}] = \boldsymbol{\mathcal{E}} \cos(\omega t). \tag{8.1}$$

The resulting Lorentz three-force on the particle is

$$\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{8.2}$$

The second term can be neglected since $v << 1$ and $B = E$ in the wave. Thus, the resulting three-acceleration is

$$\boldsymbol{a} = \frac{\boldsymbol{f}}{m} = \frac{q\boldsymbol{\mathcal{E}}}{m} \cos(\omega t). \tag{8.3}$$

Putting this into Larmor's formulae (7.17) and (7.18) and taking the time average, we get

$$\frac{dP}{d\Omega} = \frac{q^4 \mathcal{E}^2}{32\pi^2 m^2} \sin^2 \varphi, \tag{8.4}$$

$$P = \frac{q^4 \mathcal{E}^2}{12\pi m^2} \tag{8.5}$$

The incident flux of the wave is given by the time average of the Pointing vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$. Since the electric and magnetic fields are perpendicular, and have equal amplitudes,

$$F = \frac{1}{2} \mathcal{E}^2 \tag{8.6}$$

Define the *differential cross section* for scattering into angle $\varphi$ by

$$\frac{d\sigma}{d\Omega} = \frac{dP}{F d\Omega}. \tag{8.7}$$

Therefore, for electron scattering we find

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{e^4}{16\pi^2 m^2} \sin^2 \varphi, \\ &= r_0^2 \sin^2 \varphi, \end{aligned} \tag{8.8}$$

where

$$r_0 = \frac{e^2}{4\pi\varepsilon_0 m c^2} \tag{8.9}$$

is the *classical electron radius.*

Integrating over solid angle gives the total cross section

$$\sigma = \sigma_T \equiv \frac{8\pi}{3} r_0^2, \tag{8.10}$$

which is called the *Thompson cross section.*

The differential cross section for unpolarized radiation can be found by averaging around the direction of the incident radiation. Drawing a spherical triangle with vertices corresponding to the directions of the incident and outgoing waves and the electric field vector, one finds $\cos\varphi = \sin\theta\cos\phi$, so

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= r_0^2(1 - \langle \cos^2\phi \rangle \sin^2\theta), \\ &= r_0^2(1 - \frac{1}{2}\sin^2\theta), \\ &= \frac{1}{2} r_0^2 (1 + \cos^2\theta). \end{aligned} \tag{8.11}$$

In the rest frame of the particle, the incident and scattered radiation has the same frequency. Therefore, the energy of an incident and scattered photon is the same. This is an example of *coherent scattering.*

## 8.2 Compton scattering

Compton scattering occurs when the energy of the incident photon is sufficiently great that significant momentum is imparted to the charged particle. As a result, the energy of the photon is changed by the scattering process. Let $\vec{k}_i$ and $\vec{k}_f$ be the initial and final four-frequencies of the photon. Similarly, let $\vec{p}_i$ and $\vec{p}_f$ be the initial and final four-momenta of the particle. (The subscripts here are labels, not vector indices). Then conservation of four-momentum requires that

$$\vec{k}_i + \vec{p}_i = \vec{k}_f + \vec{p}_f. \tag{8.12}$$

Chose a frame in which the particle is initially at rest. Then, $\vec{p}_i = m(1, \mathbf{0})$. The photon momenta are $\vec{k}_i = \omega_i(1, \boldsymbol{n}_i)$ and $\vec{k}_f = \omega_f(1, \boldsymbol{n}_f)$, where $\boldsymbol{n}_i$ and $\boldsymbol{n}_f$ are the initial and final directions of the photons ($\hbar = 1$). Then, we have

$$\begin{aligned} m^2 = p_f^2 &= (\vec{k}_i + \vec{p}_i - \vec{k}_f)^2, \\ &= m^2 + 2\vec{p}_i \cdot (\vec{k}_i - \vec{k}_f) - 2\vec{k}_i \cdot \vec{k}_f, \\ &= m^2 + 2m(\omega_i - \omega_f) - 2\omega_i\omega_f(1 - \boldsymbol{n}_i \cdot \boldsymbol{n}_f). \end{aligned} \tag{8.13}$$

In terms of the wavelength, $\lambda = 2\pi/\omega$, this becomes

$$\lambda_f = \lambda_i + \lambda_c(1 - \cos\varphi), \tag{8.14}$$

where $\varphi$ is the angle between the initial and final photon direction and $\lambda_c = 2\pi/m = h/mc$ is the *Compton wavelength*. It is the wavelength for which $\hbar\omega = mc^2$. For an electron, $\lambda_c \sim 0.002426$ nm. Photons that have a wavelength much larger than this cannot change appreciably change the energy of the electron, so the collision corresponds to Thompson scattering. On the other hand, high-energy photons, with $\lambda << \lambda_c$ can accelerate the electron to relativistic velocity.

The cross section for Compton scattering is given by the Klein-Nishina formula, derived using quantum electrodynamics,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 \frac{\omega_f^2}{\omega_i^2} \left( \frac{\omega_i}{\omega_f} + \frac{\omega_f}{\omega_i} - \sin^2 \varphi \right) \tag{8.15}$$

This is smaller than the for Thompson scattering. Scattering is less efficient at high energies.

The total scattering cross section is

$$\sigma = \sigma_T \frac{3}{4} \left\{ \frac{1+x}{x^3} \left[ \frac{2x(1+x)}{1+2x} - \ln(1+2x) \right] + \frac{1}{2x} \ln(1+2x) - \frac{1+3x}{(1+2x)^2} \right\}, \tag{8.16}$$

where $x = \omega_i/m = \lambda_c/\lambda_i$. This is plotted for a range of $x$ in Figure (8.1).

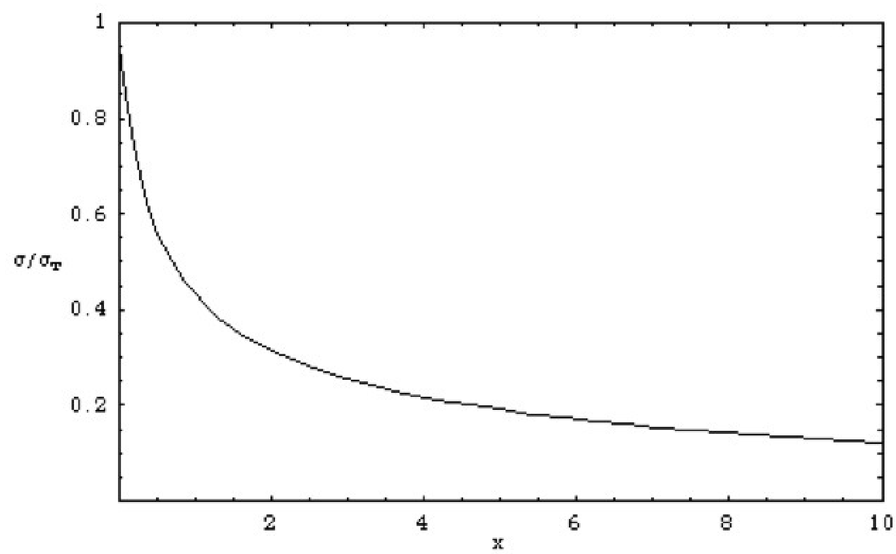



Figure 8.1: Compton scattering cross section. The figure shows the cross section, in units of the Thompson cross section, as a function of the dimensionless energy parameter $x = \omega_i/m = \lambda_c/\lambda_i$.

# 9 Inverse Compton radiation

If relativistic electrons encounter low-energy photons, Compton scattering can transfer energy from the electrons to the photons, boosting them even to gamma-ray energies. This is called *inverse Compton radiation.* In the rest frame of the electron, we have (8.14). Using primes to denote the electron rest frame, this becomes

$$\omega'_f = \frac{\omega'_i}{1 + \lambda_c \omega'_i (1 - \cos\varphi')} \tag{9.1}$$

Now transform this to the lab frame, in which the electron is moving with velocity $v \simeq 1$ in the $z$ direction. Let the initial and final photon directions in this frame be given by $(\theta_i, \phi_i)$ and $(\theta_f, \phi_f)$. These are related to the angles in the electron's rest frame by the aberration formulae, (4.43, 4.44). In the electron rest frame, the angle $\varphi'$ between these two directions can be found from spherical triangles,

$$\cos\varphi' = \cos\theta'_i \cos\theta'_f + \sin\theta'_i \sin\theta'_f \cos(\phi_f - \phi_i) \tag{9.2}$$

(the angle $\phi$ is not affected by the boost).

The initial and final photon frequencies are given by the relativistic Doppler formula (4.36). We see that the maximum energy boost occurs for a head-on collision ($\varphi' = \pi$). The photon frequency is boosted by a factor of $\sim 2\gamma$ going from the lab frame to the electron rest frame, and by another factor of $\sim 2\gamma$ going back into the lab frame, for a total boost of $\sim 4\gamma^2$.

## 9.1 Isotropic photon distribution

Lets now calculate the power produced by inverse Compton scattering. Let the intensity of incident photons be $I_\nu(\cos\theta)$. In most cases, the electrons are encountering a distribution of photons that is isotropic, or nearly so. For example, the cosmic microwave background (CMB) photons, so $I_\nu$ will be independent of direction.

In the electron rest frame, the scattering process is Thompson scattering. In this frame the incident radiation is not isotropic because of aberration and the doppler shift. Since $I_\nu/\nu^3$ is invariant, the intensity in the rest frame is $I'_\nu = I_\nu(\nu'/\nu)^3$. The power scattered, in this frame, is given by $P'$, but since Thompson scattering is symmetric, this is the same as the power $P$ in the lab frame. Thus,

$$\begin{aligned}
P = P' &= \sigma_T \int_{4\pi} d\Omega' \int_0^\infty I'_\nu(\cos\theta') d\nu', \\
&= 2\pi\sigma_T \int_0^\pi \sin\theta' d\theta' \int_0^\infty \left(\frac{\nu'}{\nu}\right)^3 \nu' I_\nu(\cos\theta) \frac{d\nu'}{\nu'}, \\
&= 2\pi\sigma_T \nu \int_{-1}^1 d\left(\frac{\cos\theta - v}{1 - v\cos\theta}\right) \int_0^\infty \gamma^4 (1 - v\cos\theta)^4 \nu I_\nu \frac{d\nu}{\nu}, \\
&= 2\pi\sigma_T \int_{-1}^1 \frac{1 - v^2}{(1 - v\cos\theta)^2} d(\cos\theta) \int_0^\infty \gamma^4 (1 - v\cos\theta)^4 I_\nu d\nu, \\
&= 2\pi\sigma_T \gamma^2 \int_{-1}^1 (1 - v\cos\theta)^2 d(\cos\theta) \int_0^\infty I_\nu d\nu.
\end{aligned} \tag{9.3}$$

For an isotropic distribution, $I_\nu$ does not depend on $\theta$ so this becomes

$$P = 2\pi\sigma_T\gamma^2 \int_{-1}^{1} (1 - vx)^2 dx \int_0^\infty I_\nu d\nu,$$
$$= 4\pi\sigma_T\gamma^2 \left(1 + \frac{1}{3}v^2\right) I,$$
$$= \sigma_T\gamma^2 U_\gamma \left(1 + \frac{1}{3}v^2\right), \tag{9.4}$$

where $U_\gamma$ is the energy density of the incident photon field. To get the net power radiated, we must subtract from this the power that is lost from the incident radiation, which is $c\sigma_t U\gamma$. Therefore,

$$P = \sigma_T U_\gamma \left[\gamma^2 \left(1 + \frac{1}{3}v^2\right) - 1\right],$$
$$= \sigma_T U_\gamma \left(\frac{1}{3}\gamma^2 v^2 + \gamma^2 - 1\right),$$
$$= \frac{4}{3}\sigma_T\gamma^2 v^2 U_\gamma. \tag{9.5}$$

## 9.2 Spectrum of the radiation

The spectrum of the radiation can be calculated using quantum field theory. For an isotropic distribution of photons of frequency $\nu_0$ and number density $n_\gamma$, and an isotropic distribution of electrons of energy $\gamma mc^2$ and number density $n_e$, the emission coefficient is given by

$$j_\nu(\gamma, \nu_0) = \frac{3\pi\hbar c}{2}\sigma_T n_e n_\gamma g(\nu/4\gamma^2\nu_0), \tag{9.6}$$

where

$$g(x) = 2x^2 \ln x + x^2 + x - 2x^3, \qquad 0 < x < 1. \tag{9.7}$$

This function is shown in Figure 9.1.

Of course we are rarely dealing with electrons having a single energy. More likely, the electrons have a distribution of energies which, over some range, can be represented by a power law, $n_e(\gamma) = n_0\gamma^{-p}$, where $n_0$ and $p$ are constants. In that case, we must integrate over $\gamma$ to get the total emission. We may also have a distribution of initial photon frequencies. Let $n_\nu(\nu_0)$ be the number of photons per unit volume per Hz having frequency $\nu_0$. Then the total emission is,

$$j_\nu = \frac{3\hbar c\sigma_T 2^{p-1}(p^2 + 4p + 11)}{(p+3)^2(p+5)(p+1)} n_0 \nu^{-(p-1)/2} \int_0^\infty n_\nu(\nu_0)\nu_0^{(p-1)/2} d\nu_0. \tag{9.8}$$

We see that the spectrum is a power law, $j_\nu \propto \nu^{-s}$ with spectral index $s = (p-1)/2$.

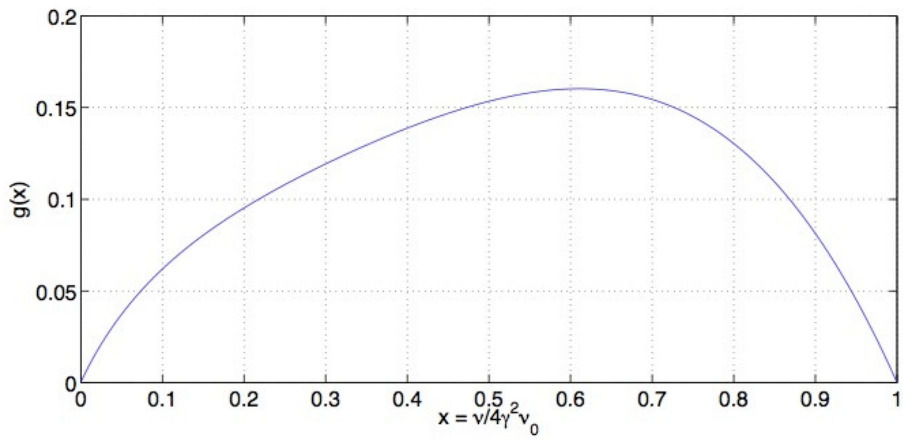



Figure 9.1: Spectrum of inverse Compton radiation, for an isotropic distribution of photons having initial frequency $\nu_0$ and electrons having energy $\gamma mc^2$.

## 9.3 The Compton y parameter

Photons passing through a medium containing free electrons repeatedly scatter. This changes the frequency spectrum of the photons. The fractional change in photon energy is described by the *Compton y parameter*,

$$y = \ln(\nu_f/\nu_i). \tag{9.9}$$

For low-energy photons traveling a distance L through a thermal distribution of electrons, one has

$$y \simeq \begin{cases} \dfrac{4T}{m}\tau(\tau+1), & \gamma \ll 1, \\ \left(\dfrac{4T}{m}\right)^2 \tau(\tau+1), & \gamma \gg 1. \end{cases} \tag{9.10}$$

where $\tau \simeq n_e \sigma_T L$ is the *optical depth*.

## 9.4 Sunyaev-Zel'dovich effect

Inverse-Compton scattering of CMB photons by electrons in the hot intergalactic medium in clusters of galaxies produces detectable changes in the CMB spectrum. This was predicted by Rachid Sunyaev and Yakov B. Zel'dovich in 1969. The *Sunyaev-Zel'dovich effect* was soon observed and provided a means of detecting distant clusters of galaxies by imaging the CMB radiation to look for spatially-resolved spectral distortions. The principal effect arises from the thermal velocities of electrons in the hot gas (several million degrees) found in rich clusters of galaxies. CMB photons are boosted in energy, resulting in a slight increase in the CBM intensity at shorter wavelengths and a decrease at longer wavelengths. The effect is independent of the distance to the cluster.

# 10 Radiative transfer

As radiation propagates through a medium, its intensity may be reduce due to absorption or scattering of photons, or increased by emission from the medium. Emission of radiation is characterized by the *emission coefficient* $j_\nu(\boldsymbol{n}$, which is the power emitted per Hz per steradian in the direction $\boldsymbol{n}$. For isotropic emission, one also defines the *emissivity* $\varepsilon_\nu$, which is the total power emitted per unit frequency per unit mass. Thus,

$$j_\nu = \frac{\varepsilon_\nu \rho}{4\pi} \tag{10.1}$$

where $\rho$ is the mass density of the medium.

As the radiation propagates, the decrease in intensity, per unit distance $s$, must be proportional to the intensity. The proportionality constant is called the *absorption coefficient* $\alpha(\nu)$.

Combining these two processes we obtain the *equation of radiative transfer*

$$\frac{dI_\nu}{ds} = -\alpha(\nu) I_\nu + j_\nu. \tag{10.2}$$

## 10.1 Optical depth

To better understand this equation, consider the case of absorption only, $j_\nu = 0$. Then

$$\frac{dI_\nu}{ds} = -\alpha(\nu) I_\nu, \tag{10.3}$$

which has the solution

$$I_\nu(s) = I_\nu(0) e^{-\tau(s)}, \tag{10.4}$$

where

$$\tau(s) = \int_0^s \alpha(\nu, x) dx \tag{10.5}$$

is called the *optical depth*. We see that if the optical depth is zero, there is no absorption. In the presence of absoprtion the optical depth increases along the propagation path. By the time that it reaches unity, a fraction $1/e$ of the radiation has been absorbed. The medium is said to be *optically-thick* if $\tau \gg 1$ and *optically-thin* if $\tau \ll 1$.

In the case of emission only, we have the solution

$$I_\nu(0) = I_\nu(s) + \int_0^s j_\nu(x) dx, \tag{10.6}$$

which shows how the emission contributes to the intensity.

### 10.2 Source function

From the definition (10.5) we see that $d\tau = \alpha(s)ds$, so (10.2) can be written as

$$\frac{dI_\nu}{d\tau} = S_\nu - I_\nu, \tag{10.7}$$

where

$$S_\nu = \frac{I_\nu}{\alpha(\nu)} \tag{10.8}$$

is called the *source function*.

The general solution of this equation can be found by multiplying both sides by the *integrating factor* $e^\tau$,

$$e^\tau \frac{dI_\nu}{d\tau} = \frac{d}{d\tau}(e^\tau I_\nu) - I_\nu e^\tau = S_\nu e^\tau - I_\nu e^\tau, \tag{10.9}$$

therefore,

$$I_\nu = I_\nu(0)e^{-\tau} + e^{-\tau}\int_0^\tau S_\nu(x)e^x dx \tag{10.10}$$

If the source function is constant along the propagation path, this reduces to

$$I_\nu = I_\nu(0)e^{-\tau} + S_\nu(1 - e^{-\tau}) \tag{10.11}$$

which shows that if the medium is optically-thick at frequency $\nu$, $I_\nu \simeq S_\nu$.

### 10.3 Mean free path

The equation of radiative transfer tells us that the probability that a photon will travel at least to an optical depth $\tau$ is $e^{-\tau}$. Therefore, the mean optical depth reached by a photon before being absorbed is

$$\langle\tau\rangle = \int_0^\infty \tau e^{-\tau} d\tau = 1. \tag{10.12}$$

If the absorption coefficient is constant, the mean distance travelled by the photon before absorption will be

$$l = \langle s\rangle = \langle\tau/\alpha\rangle = 1/\alpha. \tag{10.13}$$

This distance is called the *mean free path*.

# 11 Black body radiation

Black body radiation refers to a radiation field that is in thermal equilibrium with matter. An example is the interior of a box, or cavity, which is held at a uniform temperature T. Oscillations of electrons in the walls generate electromagnetic radiation that propagates in the interior of the box. Conversely, radiation is absorbed by interaction with the walls. If a small hole is made in the side of the box, black-body radiation leaks out from the interior. This radiation has a universal spectrum $I_\nu = B_\nu$ called the Planck function, which depends only on the temperature. To see this imagine two sources of black body radiation, at the same temperature, placed adjacent to one another with the holes aligned, and separated only by a filter that passes a narrow range of frequencies around $\nu$. If the spectra emitted by each were not identical, energy would flow from one to the other, in violation of the second law of thermodynamics.

A body that is perfectly absorbing, and has a uniform temperature $T$, must emit exactly the same power, per unit surface area, with a Planck spectrum. To see this, imagine that the body is placed inside a box filled with block body radiation. In equilibrium, every element of the surface of the body must emit as much radiation as it absorbs. Since it is a perfect absorber, the emitted flux is the same as the flux of black body radiation within the cavity, at all frequencies.

## 11.1 Kirchoff's law

Using the same argument as for a solid absorber, we could fill the black-body cavity with some medium that both absorbs and emits radiation and allow it to come to thermal equilibrium. The radiation in the cavity is homogeneous and isotropic and has the Planck spectrum. Therefore $dI_\nu/ds = 0$ and from the equation of radiative transfer we have

$$B_\nu = S_\nu = \frac{J_\nu}{\alpha}. \tag{11.1}$$

Thus, the source function of a thermal emitter is the Planck function and there is an connection between emission and absorption,

$$j_\nu = \alpha B_\nu. \tag{11.2}$$

## 11.2 Density of states

To find the Planck function, we imagine radiation filling a cube having sides of length $L$, and impose periodic boundary conditions. The components of the wave vector $\boldsymbol{k}$ must satisfy

$$\frac{k_\alpha L}{2\pi} = n_\alpha \tag{11.3}$$

where $n_\alpha = 1, 2, \cdots$. Each integer value corresponds to a different quantum state, and corresponds to a point in k-space. Each point occupies a volume $(2\pi/L)^3$ in k-space, so the total number of states contained within a volume $d^3k = k^2 dk d\Omega$ is $(L/2\pi)^3 k^2 dk d\Omega$.

We must multiply this by the number of spin states $g$ that the particle can have. For a photon there are two, corresponding to RH and LH circular polarization so $g = 2$. Thus, the number of states per unit volume is

$$\begin{aligned} dn &= (2\pi)^{-3} g k^2 dk d\Omega, \\ &= (2\pi)^{-3} g \omega^2 d\omega d\Omega. \end{aligned} \tag{11.4}$$

## 11.3 Occupation number

Photons obey Bose-Einstein statistics. There is no limit to the number of photons that can occupy a given quantum state. Suppose that there are $j$ photons in a state of frequency $\omega$. Then the energy of these photons will be $E = j\hbar\omega = j\omega$. According to the fundamental law of statistical mechanics, the probability that a system in equilibrium at temperature T will have energy $E$ is $e^{-E/k_B T}$, where $k_B$ is Boltzmann's constant ($1.38 \times 10^{-23}$ J K$^{-1}$, hereafter taken to be unity by a redefinition of the unit of temperature). Therefore, the mean number of photons will be

$$\begin{aligned} N &= \frac{1}{Z} \sum_{j=0}^{\infty} j e^{-jx}, \\ &= -\frac{1}{Z} \frac{dZ}{dx} \end{aligned} \tag{11.5}$$

where

$$Z = \sum_{j=0}^{\infty} e^{-jx}, \tag{11.6}$$

is the *partition function* and $x = \omega/T$.

Since

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \tag{11.7}$$

we have

$$Z(x) = \frac{1}{1-e^{-x}}, \tag{11.8}$$

therefore

$$N = \frac{e^{-x}}{1-e^{-x}} = \frac{1}{e^x - 1}. \tag{11.9}$$

## 11.4 Planck function

To find the energy in each state, we multiply the number of photons in that state by the corresponding frequency $\omega$, so the energy per unit volume, with frequency in the range $(\omega, \omega + d\omega)$ traveling within solid angle $d\Omega$ is

$$I_\omega d\omega d\Omega = (2\pi)^{-3} (g\omega^2 d\omega d\Omega) \frac{\omega}{e^x - 1}. \tag{11.10}$$

Recalling that $g = 2$ for photons we get the intensity

$$I_\omega = B_\omega = \frac{\omega^3}{4\pi^3}\frac{1}{e^{\omega/T}-1}, \tag{11.11}$$

$$I_\nu = B_\nu = \frac{4\pi\nu^3}{e^{2\pi\nu/T}-1}. \tag{11.12}$$

which in SI units is

$$B_\nu = \frac{2h\nu^3}{c^2(e^{h\nu/k_BT}-1)}. \tag{11.13}$$

## 11.5 Energy and number density

The specific energy density can be found immediately,

$$U_\omega = 4\pi B_\omega = \frac{\omega^3}{\pi^2(e^{\omega/T}-1)}. \tag{11.14}$$

The specific number density is

$$n_\omega = \frac{U_\omega}{\omega} = 4\pi B_\omega = \frac{\omega^2}{\pi^2(e^{\omega/T}-1)}. \tag{11.15}$$

To obtain the energy density we must integrate over frequency,

$$\begin{aligned} U &= \frac{1}{\pi^2}\int_0^\infty \frac{\omega^3 d\omega}{e^{\omega/T}-1}, \\ &= \frac{T^4}{\pi^2}\int_0^\infty \frac{x^3 dx}{e^x-1}. \end{aligned} \tag{11.16}$$

The integral has the value $6\zeta(4) = \pi^4/15$, where $\zeta(x)$ is the Riemann zeta function, defined by

$$\zeta(x) = 1 + 2^{-x} + 3^{-x} + \cdots = \frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t-1}dt, \qquad (x > 1). \tag{11.17}$$

Thus, the energy density is

$$U = \frac{\pi^2}{15}T^4 = aT^4. \tag{11.18}$$

where the radiation constant $a$ has the value in SI units

$$a = \frac{8\pi^5 k^3}{15h^3c^3} = 7.5657 \times 10^{-16}\ \mathrm{J\ m^{-3}\ K^{-4}}. \tag{11.19}$$

The number density is found in a similar manner,

$$\begin{aligned} n &= \frac{T^3}{\pi^2}\int_0^\infty \frac{x^3 dx}{e^x-1}, \\ &= \frac{2\zeta(3)}{\pi^2}T^3. \end{aligned} \tag{11.20}$$

The flux is given by

$$F = \pi I = \frac{U}{4} = \sigma_B T^4 \tag{11.21}$$

where

$$\sigma_B = \frac{2\pi^5 k_B^3}{15h^3c^2} = 5.6704 \times 10^{-8} \text{ J m}^{-2} \text{ s}^{-1} \text{ K}^{-4} \tag{11.22}$$

is the Stefan-Boltzman constant.

## 11.6 Fermions

Some relativistic particles, such as neutrinos, are fermions. These particles obey Fermi-Dirac statistics, and at most one particle can occupy a given quantum state. In that case, the partition function is

$$Z = \sum_{j=0}^{1} e^{-x} = 1 + e^{-x}, \tag{11.23}$$

so the occupation number is

$$n = \frac{e^{-x}}{1+e^{-x}} = \frac{1}{e^x+1}. \tag{11.24}$$

which differs from that of bosons by an important sign. The specific energy and number densities are therefore

$$U_\omega = \frac{g_F \omega^3}{2\pi^2 (e^{\omega/T}+1)}, \tag{11.25}$$

$$n_\omega = \frac{U_\omega}{\omega} = 4\pi B_\omega = \frac{g_F \omega^2}{2\pi^2 (e^{\omega/T}+1)}. \tag{11.26}$$

Integration gives the total densities, which we denote with an $F$ for fermion, as opposed to $B$ for boson.

$$U_F = \frac{7\pi^2}{240} T^4 = \frac{7}{8}\frac{g_F}{g_B} U_B, \tag{11.27}$$

$$n_F = \frac{3\zeta(3) g_F}{4\pi^2} T^3 = \frac{3}{4}\frac{g_F}{g_B} n_B. \tag{11.28}$$

# 12 Bremsstrahlung

Bremsstrahlung refers to radiation that is produce when a moving charge is accelerated by the coulomb field of another charge. Most commonly this occurs in an astrophysical plasma when electrons are deflected as they pass near to ions. Electrons dominate the emission because their lower mass results in greater acceleration. Bremsstrahlung is also called free-free emission, as the electron is not in a bound state.

A complete analysis requires quantum electrodynamics. However, the basic results can be obtained from a classical analysis, to which are added corrections from the quantum theory.

## 12.1 Single non-relativistic electron

Consider an electron encountering an ion (Figure 12.1). Classically, the Coulomb attraction results in a deflection of the electron, which results in radiation. Ions are several thousand times more massive than an electron, so the ion can be considered as fixed. The deflection of the electron is generally very small so the path can be approximated by a straight line (for an exact approach, see Landau and Lifshitz, *The Classical Theory of Fields*).

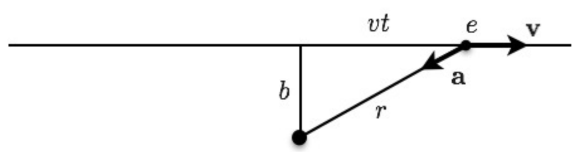



Figure 12.1: Geometry of electron-ion encounter.

Let the charge of the ion be $Ze$ and let $\boldsymbol{r}$ be the vector connecting the ion to the electron. The acceleration of the electron is given by

$$\boldsymbol{a} = -\frac{Ze^2}{4\pi m r^3}\boldsymbol{r} = -\frac{Zr_0}{r^3}\boldsymbol{r} \tag{12.1}$$

Therefore,

$$a^2 = \frac{Z^2 r_0^2}{(b^2 + v^2 t^2)^2}. \tag{12.2}$$

Putting this into Larmor's formula (7.18) we get the total power radiated as a function of time,

$$P(t) = \frac{e^2 a^2}{6\pi} = \frac{\alpha \sigma_T Z^2}{4\pi (b^2 + v^2 t^2)^2}., \tag{12.3}$$

where $\alpha = e^2/4\pi \simeq 1/137$ is the fine structure constant.

The total energy radiated is

$$\begin{aligned} W = \int_{-\infty}^{\infty} P dt &= \frac{\alpha \sigma_T Z^2}{4\pi} \int_{-\infty}^{\infty} \frac{dt}{(b^2 + v^2 t^2)^2}, \\ &= \frac{\alpha \sigma_T Z^2}{4 v b^3}. \end{aligned} \tag{12.4}$$

## 12.2 Spectrum of the radiation

The spectrum of the emitted power is related to the Fourier transform of the acceleration of the electron, which we define by

$$\tilde{a}(\omega) = \int_{-\infty}^{\infty} a(t) e^{-i\omega t} dt, \tag{12.5}$$

$$a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{a}(\omega) e^{i\omega t} d\omega, \tag{12.6}$$

Using (12.3), we can write the radiated energy as

$$W = \int_{-\infty}^{\infty} P(t) dt = \frac{e^2}{6\,\pi} \int_{-\infty}^{\infty} a^2(t) dt. \tag{12.7}$$

By Parseval's theorem, this can be written as an integral over frequency,

$$W = \frac{e^2}{12\pi^2} \int_{-\infty}^{\infty} |\tilde{a}(\omega)|^2 d\omega. \tag{12.8}$$

Since $a(t)$ is real, $\tilde{a}(\omega)$ is Hermetian (its real part is symmetric and its imaginary part is antisymmetric). Thus

$$W = \frac{e^2}{6\pi^2} \int_{0}^{\infty} |\tilde{a}(\omega)|^2 d\omega. \tag{12.9}$$

From this we see that the energy emitted per unit angular frequency is given by

$$W_\omega \equiv \frac{dW}{d\omega} = \frac{e^2}{6\pi^2} |\tilde{a}(\omega)|^2. \tag{12.10}$$

Evaluating the Fourier transform, we get

$$\begin{aligned} \tilde{a}(\omega) &= Zr_0 \int_{-\infty}^{\infty} \frac{e^{-i\omega t} dt}{b^2 + v^2 t^2}, \\ &= \frac{Zr_0}{vb} \int_{-\infty}^{\infty} \frac{e^{-i(\omega b/v)x} dx}{1 + x^2}, \\ &= \frac{\pi Z r_0}{vb} e^{-|\omega b/v|}. \end{aligned} \tag{12.11}$$

Thus,

$$W_\omega = \frac{\alpha \sigma_T Z^2}{4 v^2 b^2} e^{-2\omega b/v}. \tag{12.12}$$

We see that the spectrum of the emitted power is essentially constant when $\omega \ll v/b$.

## 12.3 Emission from many electrons

Now suppose that we have many electrons and ions, with number densities $n_e$ and $n_i$ respectively, with the electrons all having the same speed $v$. The number of collisions per unit volume, per unit time, with impact parameter in the range $(b, b+db)$ is

$$\frac{dN}{dVdt} = n_e n_i v \cdot 2\pi b db. \tag{12.13}$$

Therefore, the energy emitted per unit frequency, per unit volume, is

$$\frac{dW}{dVd\omega dt} = \frac{\pi}{2v}\alpha\sigma_T Z^2 n_i n_e \int_{b_{min}}^{b_{max}} \frac{e^{-2\omega b/v}}{b} db, \tag{12.14}$$

where $b_{min}$ and $b_{max}$ are minimum and maximum impact parameters. As a good approximation we can take $b_{max} = \infty$, however as $b_{min} \to 0$ the power diverges logarithmically. A reasonable lower limit is the distance at which quantum effects become important. From the Heisenberg uncertainty principle, $\Delta p \Delta x \sim \pi$. Taking $\Delta x = b_{min}$ and $\Delta p = mv$ gives $b_{min} \sim \pi/mv$. Inserting this into (12.12) we get

$$\frac{dW}{dVd\omega dt} = \frac{\pi\alpha\sigma_T}{2v} Z^2 n_i n_e E_1(x), ,$$

where $E_1(x)$ is an exponential integral and $x = 2\pi\omega/mv^2$.

The exact result, from quantum electrodynamics, is quite similar,

$$\frac{dW}{dVd\omega dt} = \frac{2\alpha\sigma_T}{\sqrt{3}\,v} Z^2 n_i n_e g_{ff}(v,\omega), \tag{12.15}$$

where $g_{ff}$ is a slowly-varying function of $v$ and $\omega$ called the *Gaunt factor*.

## 12.4 Thermal bremsstrahlung

Finally, we now allow a range of electron velocities. In astrophysics settings, the electrons typically have a thermal velocity distribution, given by the *Maxwell-Boltzmann distribution.* The probability the electron has a velocity $\boldsymbol{v}$, within $d^3v$ is

$$f(\boldsymbol{v})d^3v = \left(\frac{m}{2\pi T}\right)^{3/2} e^{-mv^2/2T} d^3v \tag{12.16}$$

(The normalization constant comes from the condition that the integral of $P(\boldsymbol{v})$ over the entire three-velocity space must be unity.) Therefore,

$$f(v)dv = \left(\frac{m}{2\pi T}\right)^{3/2} 4\pi v^2 \exp(-mv^2/2k)dv. \tag{12.17}$$

Averaging the velocity over this distribution,

$$\frac{dW}{dVd\omega dt} = \frac{8\pi\alpha\sigma_T}{\sqrt{3}}\left(\frac{m}{2\pi T}\right)^{3/2} Z^2 n_i n_e \int_{v_{min}}^{\infty} g_{ff} e^{-mv^2/2T} v dv. \tag{12.18}$$

where $v_{min} = \sqrt{2\omega/m}$ is the minimum velocity of an electron that has at least energy $\omega$. Setting $x = \sqrt{mv^2/2T}$ this can be written as

$$\frac{dW}{dV d\omega dt} = \frac{4m\alpha\sigma_T}{(6\pi mT)^{1/2}} Z^2 n_i n_e \, \bar{g}_{ff} e^{-\omega/T}, \tag{12.19}$$

where

$$\bar{g}_{ff} = \int_{x_{min}}^{\infty} g_{ff}(x) e^{-x} x dx \Big/ \int_{x_{min}}^{\infty} e^{-x} x dx \tag{12.20}$$

is the Gaunt factor averaged over the velocity distribution. Its value may be approximated by the following equation, in SI units,

$$\bar{g}_{ff}(T,\nu) = 14.18 + 1.91\log(T) - 1.27\log(\nu). \tag{12.21}$$

If we assume isotropic emission and divide (12.19) by $4\pi$ steradians, we get the emission coefficient

$$\begin{aligned} j_\nu &= \frac{dW}{dV d\nu dt d\Omega} = \frac{2\pi}{4\pi}\frac{dW}{dV d\omega dt}, \\ &= 2\alpha\sigma_T \left(\frac{m}{6\pi T}\right)^{1/2} Z^2 n_i n_e \bar{g}_{ff} e^{-2\pi\nu/T} \\ &= 5.4\times 10^{-40}\, T^{-1/2} Z^2 n_e n_i \bar{g}_{ff} e^{-h\nu/kT}, \end{aligned} \tag{12.22}$$

where the last line assumes SI units.

The total power emitted per unit volume is

$$\begin{aligned} \varepsilon_{ff} &= 4\pi \int_0^\infty j_\nu d\nu, \\ &= 8\pi\alpha\sigma_T \left(\frac{m}{6\pi T}\right)^{1/2} \int_0^\infty \bar{g}_{ff} e^{-2\pi\nu/T} d\nu \\ &= 4\alpha\sigma_T \left(\frac{mT}{6\pi}\right)^{1/2} Z^2 n_i n_e \bar{g}_B & (12.23) \\ &= 1.4\times 10^{-28} T^{1/2} Z^2 n_e n_i \bar{g}_B(T), & (12.24) \end{aligned}$$

where $\bar{g}_B$ is $\bar{g}_{ff}$ averaged over frequency. It is typically in the range 1.1 to 1.5, with values near 1.2 being typical.

## 12.5 Free-free absorption

Radiation can also be absorbed by an electron moving in the electric field of an ion, with a corresponding increase in the energy of the electron. This is called free-free absorption. For thermal bremsstrahlung, the emission and absorption coefficients are related by Kirchhoffs law (11.2). Therefore the absorption coefficient is,

$$\begin{aligned} \alpha_{ff} &= \frac{j_\nu}{B_\nu} = \left(\frac{e^{\omega/T}-1}{4\pi\nu^3}\right) 2\alpha\sigma_T \left(\frac{m}{6\pi T}\right)^{1/2} Z^2 n_i n_e \bar{g}_{ff} e^{-\omega/T}, \\ &= \frac{\alpha\sigma_T}{2\pi}\left(\frac{m}{6\pi T}\right)^{1/2} Z^2 n_i n_e \bar{g}_{ff} \nu^{-3}(1-e^{-h\nu/kT}), \end{aligned} \tag{12.25}$$

In the Rayleigh-Jeans (low frequency) limit, $\alpha_{ff} \propto \nu^{-2}$, so free-free absorption cuts of the spectrum at low frequencies.

# 13 Synchrotron radiation

*Synchrotron radiation* is emitted whenever relativistic electrons move in a magnetic field. It differs from *cyclotron radiation* in which the electrons are non-relativistic. In astrophysical situations, such as radio galaxies and jets, the electrons are almost always relativistic.

The motion of an electron in a magnetic field $\boldsymbol{B}$ is determined by the relativistic equation of motion

$$\frac{d\vec{p}}{d\tau} = qF \cdot \vec{u}. \tag{13.1}$$

Recall that $\vec{p} = m\gamma(1, \boldsymbol{v})$ and $\vec{u} = \gamma(1, \boldsymbol{v})$. Also, we assume that $\boldsymbol{E} = 0$. The 0 component of (13.1) gives

$$m\frac{d\gamma}{d\tau} = -q\boldsymbol{E} \cdot \boldsymbol{v} = 0, \tag{13.2}$$

which shows that $\gamma$ is constant during the motion.

The spatial component of (13.1) is therefore

$$m\frac{d\boldsymbol{v}}{d\tau} = q\boldsymbol{v} \times \boldsymbol{B} \tag{13.3}$$

which shows that the acceleration is perpendicular to the velocity. Resolving the velocity into components parallel and and perpendicular to $\boldsymbol{B}$, $\boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}$ and recalling that $dt = \gamma d\tau$ we find

$$\begin{aligned} \frac{d\boldsymbol{v}_{\parallel}}{dt} &= 0, \\ \frac{d\boldsymbol{v}_{\perp}}{dt} &= \frac{q}{\gamma m}\boldsymbol{v}_{\perp} \times \boldsymbol{B}. \end{aligned} \tag{13.4}$$

The solution corresponds to helical motion, with electrons spiraling around the field lines with orbital frequency

$$\omega_B = \frac{qB}{\gamma mc}. \tag{13.5}$$

The acceleration is $a_{\perp} = \omega_B v_{\perp}$ and $q = -e$, so from the relativistic Larmor formula (7.20), the emitted power is

$$\begin{aligned} P &= \frac{e^2}{6\pi}\gamma^4\omega_B^2 v_{\perp}^2, \\ &= \frac{e^4}{6\pi m^2}\gamma^2 B^2 v^2 \sin^2\alpha, \\ &= \frac{8\pi}{3} r_0^2 \gamma^2 v^2 B^2 \sin^2\alpha, \\ &= \sigma_T \gamma^2 v^2 B^2 \sin^2\alpha, \end{aligned} \tag{13.6}$$

where $\alpha$ is the *pitch angle*, the angle between $\boldsymbol{v}$ and $\boldsymbol{B}$.

If the velocity distribution of the electrons is isotropic, we can average over the pitch angle to obtain $\left\langle \sin^2\alpha \right\rangle = 2/3$, thus

$$\begin{aligned} P &= \frac{2}{3}\sigma_T\gamma^2 v^2 B^2, \\ &= \frac{4}{3}\sigma_T\gamma^2 v^2 U_B, \end{aligned} \tag{13.7}$$

where $U_B = B^2/2$ is the energy density of the magnetic field. Compare this with the corresponding result for inverse Compton radiation (9.5).

## 13.1 Spectrum of the radiation

For a single electron, the spectrum of the emitted radiation, in the lab frame, has the form (see Rybiki and Lightman)

$$P_\omega = \frac{\sqrt{3}}{8\pi^2}\frac{e^3 B\sin\alpha}{m}F(\omega/\omega_c), \tag{13.8}$$

where $\omega_c = 2\pi\gamma^3\omega_B\sin\alpha$ is a characteristic frequency and the dimensionless function $F$ is given by

$$F(x) = x\int_x^\infty K_{5/3}(\xi)d\xi, \tag{13.9}$$

where $K_{5/3}$ is the modified Bessel function of order 5/3.

However, in astrophysics we almost always are dealing with an ensemble of electrons having a range of energies. Quite often, the energy distribution follows a power law, over some range of energies. As we did for inverse-Compton radiation, we write $n_e(\gamma) = n_0\gamma^{-p}$ and integrate over the distribution. This gives the emission coefficient,

$$\begin{aligned} j_\nu &= \frac{P_\omega}{4\pi} = \frac{P_\omega}{2}, \\ &= \frac{\sqrt{3}}{8\pi^2}\frac{n_0 e^3 B\sin\alpha}{m(p+1)}\Gamma\left(\frac{p}{4}+\frac{19}{2}\right)\Gamma\left(\frac{p}{4}-\frac{1}{12}\right)\left(\frac{2\pi m\nu}{3eB\sin\alpha}\right)^{(1-p)/2}, \end{aligned} \tag{13.10}$$

where $\Gamma$ is the gamma function. As for inverse Compton radiation, we see that the spectrum is a power law $j_\nu \propto \nu^{-s}$ with spectral index $s = (p-1)/2$.

## 13.2 Polarization

Because the acceleration of the electron is perpendicular to the magnetic field, and the magnetic field generally has a coherent structure over large scales, synchrotron radiation is polarized. The net polarization is linear and aligned with the magnetic field. Let $P_{\omega\parallel}$ and $P_{\omega\perp}$ be the spectral power emitted by electrons with a single energy $\gamma m$. with polarization parallel and perpendicular to the projection of the magnetic field on the plane of the sky (ie. the plane perpendicular to the line of sight). It can be shown (see Rybicki and Lightman) that

$$P_{\omega\parallel} = \frac{\sqrt{3}}{16\pi^2}\frac{e^3 B\sin\alpha}{m}\left[F(\omega/\omega_c) - G(\omega/\omega_c)\right], \tag{13.11}$$

$$P_{\omega\perp} = \frac{\sqrt{3}}{16\pi^2}\frac{e^3 B\sin\alpha}{m}\left[F(\omega/\omega_c) + G(\omega/\omega_c)\right], \tag{13.12}$$

where $F(x)$ is given by (13.9) and

$$G(x) = xK_{2/3}(x). \tag{13.13}$$

The degree of linear polarization is given by

$$P = \frac{P_{\omega\perp} - P_{\omega\parallel}}{P_{\omega\perp} + P_{\omega\parallel}} = \frac{G(\omega/\omega_c)}{F(\omega/\omega_c)}. \tag{13.14}$$

For a power-law distribution of energies, the polarization can be found by integrating over $\gamma$. Since $\omega/\omega_c \propto \gamma^{-2}$,

$$\begin{aligned} P &= \int_0^\infty G(\omega/\omega_c)\gamma^{-p}d\gamma \Big/ \int_0^\infty F(\omega/\omega_c)\gamma^{-p}d\gamma, \\ &= \int_0^\infty G(x)x^{(p-3)/2}dx \Big/ \int_0^\infty F(x)x^{(p-3)/2}dx, \\ &= \frac{p+1}{p+7/3}. \end{aligned} \tag{13.15}$$

The degree of polarization of synchrotron radiation is often quite high, sometimes exceeding 50%.

## 13.3 Self absorption

Electrons orbiting in a magnetic field can also absorb photons. For a power-law distribution of electron energies, the absorption coefficient is given by (Rybicki and Lightman),

$$\alpha(\nu) \simeq \frac{\sqrt{3}e^3 n_0 B\sin\alpha}{32\pi^2 m}\left(\frac{3eB\sin\alpha}{2\pi m^3}\right)^{p/2}\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{3p+22}{12}\right)\nu^{-(p+4)/2}, \tag{13.16}$$

which increases rapidly as frequency decreases. Comparing this to the emission coefficient (13.10), we see that the source function

$$S_\nu = \frac{j_\nu}{\alpha} \propto \nu^{5/2}. \tag{13.17}$$

Recall that when the medium is optically-thick, $I_\nu = S_\nu$. Thus, at low frequencies the spectrum increases with frequency as a power law with slope 5/2. At some frequency the medium become optically thin and the spectrum turns over to become a power law with slope $-s$. The 5/2 slope at low frequencies is a distinguishing feature of synchrotron radiation.

# 14 Radiative transitions

Emission and absorption lines arise from transitions between quantum states in atoms or molecules. Atomic lines arise from transitions of an electron between two bound states of different energy. Molecular lines arise from transitions between vibrational or rotational states, resulting in a change of vibrational or rotational energy.

A photon can trigger a transition from a lower to higher energy level if the difference in energy $E_{21} = E_2 - E_1$ is sufficiently close to the photon energy $\omega$. A *spontaneous* downward transition can occur, with the emission of a photon of energy $\omega \simeq E_{21}$. Also, *stimulated* transitions can occur if the atom or molecule is illuminated by radiation of the appropriate energy. In that case, the emitted photon is identical to the illuminating photon, having the same wavelength, direction, and phase.

As a result of this, radiation passing through gas is absorbed at specific wavelengths, giving rise to *absorption lines*. These are common in stellar spectra, due to absorption of light from the photosphere as it passes through the stellar atmosphere. Also, gas emits radiation at specific wavelengths. These result in *emission lines*, as are commonly seen in the spectra of planetary nebula and HII regions.

*Hyperfine transitions* involve the magnetic interaction between electron spin and nuclear spin. The best known is the 21-cm line of neutral hydrogen, in which transitions occur between states in which the electron and proton spins are parallel (higher energy) or anti-parallel (lower energy).

Quantum-mechanical *selection rules* govern the transitions that are allowed between different levels. The strongest lines are generally due to electric-dipole transitions, which involve a change in the electric dipole moment of the atom. For a single-electron, the selection rules for electric dipole transitions are

$$\Delta l = \pm 1, \tag{14.1}$$

$$\Delta m = 0, \pm 1. \tag{14.2}$$

where $l$ and $m$ are the orbital and magnetic quantum numbers. For multi-electron atoms we also have the total orbital and spin angular momentum quantum numbers $L$ and $S$. and the total angular momentum quantum number $J$. If the total spin angular momentum of the nucleus is included, we have the quantum number $F$, which is the total angular momentum $J$ of the electronic configuration plus the spin angular momentum of the nucleus.

For all transitions, $J$ (or $F$) can change only by 0 or $\pm 1$. The exception is $J = 0$ to $J = 0$ which is *not* allowed since the photon caries one unit of angular momentum. The selection rules of electric dipole transitions are

$$\Delta S = 0, \tag{14.3}$$

$$\Delta L = 0, \pm 1, \tag{14.4}$$

$$\Delta J = 0, \pm 1, \quad (\text{except } J = 0 \text{ to } J = 0) \tag{14.5}$$

For higher-order transitions, parity is conserved, and for magnetic dipole transitions (eg. hyperfine transitions) the electronic configuration does not change.

## 14.1 Einstein coefficients

In a semi-classical analysis of black body radiation, Einstein (1916) introduced the coefficients $B_{12}$, $B_{21}$ and $A_{21}$ to describe radiative excitation and de-excitation. The coefficient $B_{12}$ is the probability per unit time that the radiation having energy density $U_\omega(\omega)$ will cause a transition $1 \rightarrow 2$ with the absorption of a photon of energy $\omega = E_{21}$. Similarly, $B_{21}$ is the probability per unit time that the radiation having energy density $U_\omega(\omega)$ will cause a transition $2 \rightarrow 1$ with emission of a photon of energy $\omega$. The coefficient $A_{21}$ is the probability per unit time that a system in state 2 will spontaneously decay to state 1, emitting a photon.

Thus, if there are $n_1$ atoms per unit volume in state 1, the rate of absorption, per unit volume, is $U_\omega B_{12} n_1$. Similarly, the rate of emission from state 2 is $(U_\omega B_{21} n_2 + A_{21}) n_2$.

In a two-level atom (where there are no transitions involving other states), the following *rate equations* must therefore hold,

$$\frac{dn_1}{dt} = U_\omega(-B_{12}n_1 + B_{21}n_2) + A_{21}n_2, \tag{14.6}$$

$$\frac{dn_2}{dt} = U_\omega(B_{12}n_1 - B_{21}n_2) - A_{21}n_2, \tag{14.7}$$

The Einstein coefficients are related, as can be seen as follows. Place a two-level atom inside a black body cavity at temperature T and let it come to equilibrium. Then, the number of photons emitted per unit time must equal the number absorbed. Thus,

$$U_\omega(B_{21}n_2 - B_{12}n_1) + A_{21}n_2 = 0. \tag{14.8}$$

In thermal equilibrium, the ratio of populations is given by the *Boltzmann Equation*

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-E_{21}/T} = \frac{g_2}{g_1} e^{-\omega/T}, \tag{14.9}$$

which follows directly from the fundamental law of statistical mechanics. Here the *statistical weight* $g_k$ is the number of quantum states that have the same energy $E_k$.

Also, the energy density is given by the Planck function (11.14),

$$U_\omega = \frac{\omega^3}{\pi^2(e^{\omega/T} - 1)}. \tag{14.10}$$

Combining these equations, we obtain

$$\frac{\omega^3}{\pi^2(e^{\omega/T} - 1)} = \frac{A_{21}/B_{21}}{(g_1B_{12}/g_2B_{21})e^{\omega/T} - 1}, \tag{14.11}$$

which can only be true if

$$B_{21} = \frac{g_1}{g_2} B_{12}, \tag{14.12}$$

$$A_{21} = \frac{\omega^3}{\pi^2} B_{21}. \tag{14.13}$$

Note: several different definitions of the Einstein $B$ coefficients are in use. Some define the transition probability per unit time as $B_{12}U_\nu$, others as $B_{12}J_\nu$. The relationship between $A_{21}$ and $B_{21}$ will depend on which definition is used.

## 14.2 Oscillator strengths

In astronomical spectroscopy one often encounters *oscillator strengths* which are dimensionless numbers proportional to the probability of the transition. The absorption oscillator strength $f_{12}$ is calculated using quantum mechanics and is related to the Einstein coefficients by

$$B_{12} = \frac{2\pi^2 r_0}{\omega} f_{12}, \tag{14.14}$$

$$B_{21} = \frac{2\pi^2 r_0 g_1}{\omega g_2} f_{12}, \tag{14.15}$$

$$A_{21} = \frac{2 r_0 \omega^2 g_1}{g_2} f_{12}. \tag{14.16}$$

## 14.3 Line profiles

Emission and absorption lines have a finite width, due to a number of factors. *natural broadening* occurs as a result of the Heisenberg uncertainty principle. The probability that an atom in the upper state will decays to a lower state in time $t$ is

$$f(t) = e^{-\Gamma t}, \tag{14.17}$$

where the spontaneous decay rate, also called the *damping constant*, $\Gamma$ is the sum of the Einstein $A$ coefficients over all possible transitions to states of lower energy,

$$\Gamma = \sum_k A_j k, \quad E_k < E_j. \tag{14.18}$$

Therefore, the average lifetime in the upper state is $\bar{t} = 1/\Gamma$. Because of this the uncertainty in the energy is $\Delta E = \pi/\bar{t}$, so the line width is $\Delta\omega \simeq \pi\Gamma$.

The line profile can be obtained classically, by imagining the electric field to arise from a damped oscillator which decays according to

$$E \propto e^{-\frac{1}{2}\Gamma t} \tag{14.19}$$

This gives the *Lorentz line profile*

$$\begin{aligned} f(\omega) &\propto \int_0^\infty e^{-\frac{1}{2}\Gamma t} e^{-i\omega t} dt, \\ &= \frac{\Gamma}{2\pi[\omega^2 + (\Gamma/2)^2]}. \end{aligned} \tag{14.20}$$

The normalization constant was chosen so that

$$\int_{-\infty}^{\infty} f(\omega) d\omega = 1. \tag{14.21}$$

Convolving the Lorentz profile with the un-broadened line, represented by a delta function at frequency $\omega_0$, we obtain the line profile

$$\varphi_\omega = \frac{\Gamma/2\pi}{(\omega - \omega_0)^2 + (\Gamma/2)^2}, \tag{14.22}$$

which has the normalization

$$\int_{-\infty}^{\infty} \varphi_\omega d\omega = 1. \tag{14.23}$$

Several other physical process contribute to line broadening. *Pressure broadening* occurs due to collisions between atoms. Collisions can cause de-excitation of electrons, which give up their excess energy in the collision instead of emitting a photon. This reduces the effective lifetime of the excited state, increasing the linewidth. As a result, we see a Lorentzian profile whose width increases with pressure.

Another common effect is *Doppler broadening*. Typically, atoms have a thermal distribution of velocities, so the component of velocity along the line of sight (the $z$ axis say) has the distribution function

$$f(v_z) = \sqrt{\frac{m}{2\pi T}} e^{-mv^2/2T}. \tag{14.24}$$

The Doppler effect produces a frequency shift

$$\delta\omega = \omega_0 v_z, \tag{14.25}$$

so the distribution function of $\delta\omega$ is

$$f(\delta\omega) = \left| \frac{dv_z}{d(\delta\omega)} \right| f(v_z) = \frac{1}{\sqrt{\pi}\Delta\omega_D} e^{-(\delta\omega)^2/\Delta\omega_D^2}. \tag{14.26}$$

where the *Doppler width* is defined as

$$\Delta\omega_D = \omega_0 \sqrt{2T/m}. \tag{14.27}$$

This gives the Gaussian line profile

$$\varphi_\omega = \frac{1}{\sqrt{\pi}\Delta\omega_D} e^{-(\omega-\omega_0)^2/\Delta\omega_D^2}. \tag{14.28}$$

Often, spectral lines are broadened by a combination of these effects, and can be represented by a Voigt profile, which is a convolution of Gaussian and Lorentz profiles.

$$\varphi_\omega = \frac{1}{\sqrt{\pi}\Delta\omega_D} V(T/4\Delta\omega_D, (\omega - \omega_0)/\Delta\omega_D) \tag{14.29}$$

where

$$V(a, u) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{a^2 + (u - y)^2}. \tag{14.30}$$

### 14.4 Relation to emission and absorption coefficients

It is conventional to combine stimulated emission with absorption, since both depend on the energy density of the radiation. Emission is then is then included with absorption as contributing with a minus sign (ie. a negative absorption coefficient). Thus, the emission coefficient depends only on the spontaneous emission. The power emitted per steradian per unit frequency per unit volume is

$$j_\omega(\omega) = \frac{\hbar\omega}{4\pi} n_2 A_{21} \varphi_\omega, \tag{14.31}$$

$$j_\nu(\nu) = \frac{\hbar\nu}{2} n_2 A_{21} \varphi_\nu, \tag{14.32}$$

$$\tag{14.33}$$

The power absorbed from a beam per steradian per Hz per unit area per unit distance is

$$\alpha(\omega) J_\omega = \frac{\omega}{4\pi}(n_1 B_{12} - n_2 B_{21})\varphi_\omega U_\omega, \tag{14.34}$$

so the absorption coefficient is

$$\alpha(\omega) = \hbar\omega(n_1 B_{12} - n_2 B_{21})\varphi_\omega, \tag{14.35}$$

$$\alpha(\nu) = \hbar\nu(n_1 B_{12} - n_2 B_{21})\varphi_\nu. \tag{14.36}$$

### 14.5 Masers

If the matter is in thermal equilibrium with itself (but not necessarily with the radiation), the Boltzmann equation tells us that

$$\frac{n_2 B_{21}}{n_1 B_{12}} = \frac{n_2 g_1}{n_1 g_2} = e^{-\omega/T}, \tag{14.37}$$

so

$$\alpha(\omega) = \omega\varphi_\omega(n_1 B_{12}(1 - e^{\omega/T}). \tag{14.38}$$

Clearly, the first term dominates, so the absorption coefficient is positive.

If, however,

$$\frac{n_2}{n_1} \neq \frac{g_2}{g_1} = e^{-\omega/T}, \tag{14.39}$$

the system is said to be producing *non-thermal* radiation. If $n_2 g_1 > n_1 g_2$ we call this a *population inversion*. The absorption coefficient can then become negative and stimulated emission dominates absorption. The intensity then increases exponentially as the beam propagates. Such a system is called a *maser* (microwave amplification by stimulated emission of radiation), or a *laser*. Astrophysical masers have been observed in massive interstellar clouds.

# 15 Absorption and scattering

In a scattering process, photons are redirected. If the frequency is not changed we call it *coherent scattering*. In the following we assume *isotropic scattering*, for which there is no preferred scattering direction. In addition to a term describing the decrease in intensity due to scattering, we also need a term describing an increase in intensity due to scattering of other photons into the beam. This latter term is proportional to the mean intensity.

The transfer equation becomes

$$\frac{dI_\nu}{ds} = -(\alpha + \sigma)I_\nu + \sigma J_\nu + j_\nu. \tag{15.1}$$

where $\sigma$ is the scattering coefficient. For a thermal emitter, $j_\nu = \alpha B_\nu$, so we have

$$\frac{dI_\nu}{ds} = \alpha(B_\nu - I_\nu) + \sigma(J_\nu - I_\nu). \tag{15.2}$$

We can write this in a simpler form by redefining the source function

$$S_\nu = \frac{\alpha B_\nu + \sigma J_\nu}{\alpha + \sigma}. \tag{15.3}$$

This gives

$$\frac{dI_\nu}{ds} = (\alpha + \sigma)(S_\nu - I_\nu). \tag{15.4}$$

The sum of absorption and scattering coefficients $\alpha + \sigma$ is called the *extinction coefficient*. When scattering is included, the mean free path becomes

$$l = \frac{1}{\alpha + \sigma} \tag{15.5}$$

Note that in free space ($j_\nu \simeq 0$) the source function is generally less than the Planck function, $S_\nu = \alpha B_\nu/(\alpha + \sigma)$. Lets write the source function in the form ?

$$S_\nu = \varepsilon B_\nu + (1 - \varepsilon)J_\nu. \tag{15.6}$$

where

$$\varepsilon = \frac{\alpha}{\alpha + \sigma} \tag{15.7}$$

is the probability that an encounter will result in absorption The quantity $1 - \varepsilon$ is called the single-scattering *albedo*.

The typical number of scatterings that will occur before absorption will be $N = 1/\varepsilon$ , so the typical distance traveled by a photon before absorption will be

$$l_* = \sqrt{N} l = \frac{l}{\sqrt{\varepsilon}}. \tag{15.8}$$

which is called the *effective mean free path*.

The effective optical thickness, for a distance $L$, is

$$\tau_* = \frac{L}{l_*} \tag{15.9}$$

(recall that $\tau = \alpha L = L/l_*$ if $\alpha$ is constant).

## 15.1 Rayleigh scattering

Classically, an atom or molecule that has a dipole moment can be thought of as a harmonic oscillator, with some natural frequency $\omega_0$. Radiation of frequency $\omega$ incident on the atom interacts with the dipole moment, exciting the oscillator (imagine an electron attached by a spring to an atom). The oscillating dipole moment radiates at the same frequency as the incident radiation. This corresponds to a form of coherent scattering, first investigated by Lord Rayleigh.

The equation for the time evolution of the dipole moment under excitation by incident radiation is that of a driven harmonic oscillator,

$$\ddot{d} + \omega_0^2 d = \frac{e^2\mathcal{E}}{m} cos(\omega t). \tag{15.10}$$

Writing $d(t) = ex(t) = ex_0 cos\omega t$, this becomes

$$(-\omega^2 + \omega_0^2)x_0 = \frac{e\mathcal{E}}{m}. \tag{15.11}$$

Therefore the oscillation amplitude is

$$x_0 = \frac{e^2\mathcal{E}}{4\pi m}\frac{1}{\omega_0^2 - \omega^2}, \tag{15.12}$$

and the acceleration is

$$\boldsymbol{a} = \frac{e^2\boldsymbol{\mathcal{E}}}{4\pi m}\frac{\omega^2}{\omega_0^2 - \omega^2}. \tag{15.13}$$

Apart from the factor of $(\omega^2/(\omega_0^2 - \omega^2)$, this is the same as for Thompson scattering, Therefore, the scattering cross-section is the same as for Thompson scattering multiplied by the square of this factor,

$$\frac{d\sigma_R}{d\Omega} = r_0^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2} \sin^2 \varphi. \tag{15.14}$$

$$\sigma_R = \sigma_T \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}. \tag{15.15}$$

$$\tag{15.16}$$

As for Thompson scattering, the scattered light is strongly polarized. If the incident radiation is unpolarized,

$$\frac{d\sigma_R}{d\Omega} = \frac{1}{2} r_0^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}(1 + \cos^2\theta). \tag{15.17}$$

We can now generalize this to a medium containing $n$ scattering electrons per unit volume. The scattering coefficient, per unit volume, is

$$k_R = n_e \sigma_R = \frac{n_e \sigma_T \omega^4}{(\omega_0^2 - \omega^2)^2}. \tag{15.18}$$

In the low-frequency limit this reduces to Rayleigh's scattering formula

$$k_R = n_e \sigma_R = \frac{n_e \sigma_T \lambda_0^4}{\lambda^4}. \tag{15.19}$$

and in the high-frequency limit, it reduces to Thompson's formula

$$k_T = n_e \sigma_T. \tag{15.20}$$

Rayleigh scattering is the main reason that the sky is blue and sunsets are red.

## 15.2 Mie scattering

Light is also scattered by small solid or dielectric particles, called *dust grains* or *aerosols*. This is called *Mie scattering*. The scattering cross-section can be found by solving Maxwell's equations for a dielectric sphere illuminated by an incident plane wave. The result depends on the size of the particle, compared to the wavelength and its dielectric constant.

Consider the case of dielectric spheres of radius $R$ and index of refraction $n$. Define

$$x = 2\pi R/\lambda \tag{15.21}$$

and the *scattering efficiency factor* $Q_s = \sigma_s/\sigma_g$, where $\sigma_s$ is the scattering cross-section and $\sigma_g$ is the geometrical cross-section of the particle. Then

$$Q_s = \begin{cases} \simeq 0 & \text{when } x \ll 1, \\ \simeq 3 & \text{when } x \simeq 1, \\ 2 & \text{when } x \gg 1. \end{cases} \tag{15.22}$$

The last case is a consequence of *Babinet's principle*.

Mie scattering is the dominant process responsible for extinction by dust grains in the interstellar medium. These grains typically have sizes of a few microns, so at optical wavelengths the scattering efficiency decreases with increasing wavelength. Thus red light suffers less exctinction than blue which leads to *reddening* of the spectrum. A useful approximation is that

$$E_{B-V} \simeq 3A_V \tag{15.23}$$

where the reddening $E_{B-V} = (B-V) - (B-V)_0$ is the difference between the $B-V$ colour index that is observed and the value with no extinction. $A_V$ is the extinction in magnitudes in the $V$ band. In this way it is possible to estimate the extinction, from the amount of reddening of the spectrum. The latter can be deduced by comparing the spectral energy distribution of a star with the spectral type determined from line strengths.

## 15.3 Radiative diffusion

Inside stars, the effective optical depth is large and energy diffuses outward by repeated absorption, emission and scattering of photons. Lets estimate the flux flowing outward through the star. For simplicity we consider a small enough region that curvature can be ignored and approximate the medium as *plane-parallel* with all physical properties of the medium being a function of height z alone. For a ray traveling at angle $\theta$ we have $ds = dz/\cos\theta = dz/x$ where we have defined $x = \cos\theta$. The transfer equation can now be written in the form

$$I_\nu = S_\nu - \frac{x}{\alpha+\sigma}\frac{\partial I_\nu}{\partial z}. \tag{15.24}$$

The second term is generally small as the intensity changes very little over a distance comparable to the mean free path $l = 1/(\alpha+\sigma)$. So, to first order we can write $I_\nu \simeq S_\nu \simeq J_\nu$. The latter equality follows because $S_\nu$ is not a function of direction. Substituting this into the transfer equation gives $I_\nu \simeq S_\nu \simeq B_\nu$. We now have a more accurate, second order, approximation

$$I_\nu \simeq B_\nu - \frac{x}{\alpha+\sigma}\frac{\partial B_\nu}{\partial z}. \tag{15.25}$$

Now compute the specific flux

$$\begin{aligned} F_\nu(z) &= \int_{4\pi} I_\nu \cos\theta d\Omega, \\ &= 2\pi\int_{-1}^{1} B_\nu x dx - \frac{2\pi}{\alpha+\sigma}\frac{\partial B_\nu}{\partial z}\int_{-1}^{1} x^2 dx, \\ &= -\frac{4\pi}{3(\alpha+\sigma)}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z}. \end{aligned} \tag{15.26}$$

To compute the total flux, integrate over all frequencies to obtain

$$F(z) = -\frac{4\pi}{3\alpha_R}\frac{\partial T}{\partial z}\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu, \tag{15.27}$$

where

$$\alpha_R = \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu \Bigg/ \int_0^\infty \frac{1}{\alpha+\sigma}\frac{\partial B_\nu}{\partial T} d\nu, \tag{15.28}$$

is a weighted average of the extinction called the *Rosseland mean opacity.*

The integral in (15.27) can be done as follows

$$\begin{aligned} \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu &= \frac{\partial}{\partial T}\int_0^\infty B_\nu d\nu, \\ &= \frac{\partial}{\partial T}\frac{\sigma_B T^4}{\pi} = \frac{4\sigma_B T^3}{\pi}. \end{aligned} \tag{15.29}$$

Thus we obtain

$$F(z) = -\frac{16\sigma_B T^3}{3\alpha_R}\frac{\partial T}{\partial z}. \tag{15.30}$$

Equation (15.30) is called the equation of radiative diffusion and plays a key role in theoretical models of stellar structure.

## 15.4 Ionization

We have seen that excitation of atoms is governed by the Boltzmann equation. A related equation, the *Saha equation* governs the ionization state of atoms. If $n_j(X_r)$ is the number of atoms, per unit volume, that are in ionization state $r$ (have lost $r$ electrons), it follows from the fundamental equation of statistical mechanics that

$$\frac{n(X_{r+1})n_e}{n(X_r)} - \frac{Z_{r+1}Z_e}{Z_r}, \tag{15.31}$$

where $Z_r$ is the partition function for the atom in ionization state $r$,

$$Z_r = \sum_j g_{rj} e^{-E_{rj}/T} \tag{15.32}$$

and $Z_e$ is the partition function of free electrons, per unit electron density $n_e$.

Since the electron's position and momentum can vary continuously, its partition function is expressed as an integral

$$Z_e = \frac{2}{h^3} \int e^{-E/T} d^3p d^3x \tag{15.33}$$

where the factor of two enters because two electrons can occupy the same phase space volume. Dividing by the volume gives the partition function

$$\begin{aligned} Z_e &= \frac{2}{h^3} \int e^{-p_x^2/2mT} dp_x \int e^{-p_y^2/2mT} dp_y \int e^{-p_z^2/2mT} dp_z \\ &= \frac{1}{4\pi^3} (2\pi m T)^{3/2} \simeq 4.83 \times 10^{21} T^{3/2}. \end{aligned} \tag{15.34}$$

# 16 Hydrodynamics

Much of astrophysics involves the behaviour of gas, either in a neutral or ionized form. Sometimes the gas is in thermal equilibrium, as in most stars, but in other situations such as supernovae, outflows from AGN, pulsar magnetospheres, jets, etc, it is far from equilibrium. Often, magnetic fields play an important role in the dynamics of the gas. Often, relativistic motion is involved. Understanding the dynamics of gas is therefore an essential part of astrophysics.

We begin this topic by developing the most-important equations that govern gas dynamics. Unlike many other treatments, we shall start with the relativistic forms of the equations, then obtain the three-dimensional non-relativistc equations as a limit. We will make simplifying assumptions as needed to keep the results tractable, with the aim of allowing the underlying physics to emerge.

## 16.1 Dynamics of a perfect fluid

A perfect fluid is one for which molecular interactions can be ignored. Thus it has no viscosity. Such a fluid is has two characteristic scalars: the density of mass-energy $\rho$ and an isotropic pressure $P$, both defined in the local rest frame of the fluid.

The fundamental relativistic entity that embodies the fluid properties is the energy momentum tensor. The 00 component of this tensor is the mass-energy density. In the rest frame of the fluid, we must therefore have $T^{00} = \rho$. The components $T^{0\alpha}$ represent the flux of momentum, which in the rest frame must be zero. Finally, the components $T^{\alpha\beta}$ constitute the three-dimensional stress tensor, being the $\alpha$ force per unit area acting on an infinitesimal surface that is normal to the $\beta$ direction. Since there can be no shear forces in a perfect fluid, this tensor must be diagonal in the rest frame, with components equal to the isotropic pressure.

The energy-momentum tensor of a perfect fluid must therefore have the form (in the rest frame)

$$T^{jk} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{16.1}$$

We can obtain the covariant form, for an arbitrary frame, by noting that in the rest frame of the fluid, the four-velocity has the form $u = (1, 0, 0, 0)$. Therefore, the components derived above can be written as

$$T^{jk} = (\rho + P)u^j u^k - P\eta^{jk}. \tag{16.2}$$

## 16.2 Euler equations

All energy-momentum tensors are symmetric and satisfy a conservation law. Conservation of mass-energy is embodied in the requirement that the 4-divergence vanish,

$$\partial_k T^{jk} = 0. \tag{16.3}$$

This conservation law leads us to the dynamical equations for the gas.

Taking inner product of $\vec{u}$ with the divergence of (16.2) gives a scalar,

$$u^j \partial_k[(\rho + P)u^k] + (\rho + P)u^k \partial_k u^j - \partial^j P = 0. \tag{16.4}$$

Now multiply this equation by $u_j$ and sum over $j$ and note that, since $u_j u^j = 1$,

$$u_j \partial_k u^j = \frac{1}{2}\partial_k(u_j u^j) = 0 \tag{16.5}$$

Therefore,

$$\partial_k[(\rho + P)u^k] - u_j \partial^j P = 0. \tag{16.6}$$

This can be further simplified by noting that $u_j \partial^j = d/d\tau$. (To prove this, show that it is true in the rest frame. Since it is a scalar it is therefore true in all frames.) Thus we obtain the four-dimensional *continuity equation*

$$\partial_k[(\rho + P)u^k] = \frac{dP}{d\tau} \tag{16.7}$$

If the fluid motion is non-relativistic, $P \ll \rho c^2$ and $d\tau \simeq dt$ so this reduces to the three-dimensional continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \boldsymbol{v} = 0. \tag{16.8}$$

The *projection tensor* $\mathcal{P}_{ij} = \eta_{ij} - u_i u_j$ projects out the components that are orthogonal to $\vec{u}$. Multiply this by the divergence of $T^{jk}$ and sum over the index $j$ (this is called *contraction*),

$$\begin{aligned} (\eta_{ij} - u_i u_j)\left\{u^j \partial_k[(\rho + P)u^k] + (\rho + P)u^k \partial_k u^j - \partial^j P\right\} &= 0, \\ (\rho + P)u^k \partial_k u_i - (\partial_i - u_i u_j \partial^j)P &= 0. \end{aligned} \tag{16.9}$$

Thus we obtain the *relativistic Euler equation*,

$$(\rho + P)\frac{du^i}{d\tau} = \partial^i P - u^i \frac{dP}{d\tau}, \tag{16.10}$$

where

$$\frac{d}{d\tau} = u^j \partial_j = \gamma\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) = \gamma \frac{d}{dt}. \tag{16.11}$$

$d/dt$ is called the *convective derivative* and represents the rate of change seen by an observer moving with the fluid.

For a nonrelativistic fluid the terms involving both $P$ and $u^j$ are much smaller than the term involving $\rho$ and $u^k$, so this reduces to the three-dimensional Euler equation

$$\rho\frac{d\boldsymbol{v}}{dt} \equiv \rho\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = -\nabla P \tag{16.12}$$

(note the change in sign, since $\partial^\alpha = -\partial_\alpha$).

The Euler and continuity equations describe the dynamics of the fluid, showing that the acceleration of a fluid element is driven by the pressure gradient. The Euler equation is nonlinear in $\boldsymbol{v}$, making it impossible to solve in general, and giving rise to a rich set of nonlinear phenomena such as turbulence.

## 16.3 including gravity

To include gravity, one replaces partial derivatives with covariant derivatives in curved space-time. However, if the gravitational field is not strong, a Newtonian approximation will suffice. The Newtonian gravitational field $\Phi$ is defined as (minus) the energy per unit mass required to remove a fluid element to infinity. This is not a Lorentz scalar, so we define it as the gravitational potential in the rest-frame of the fluid. In natural units $\Phi$ is dimensionless and, for weak gravitational fields, it is much smaller than unity.

We must now replace the Minkowski tensor $\eta_{jk}$ by the *metric tensor* $g_{jk} = \eta_{jk} + 2\Phi\bar{\delta}_{jk}$, which in the weak-field limit, has components

$$g_{jk} = \begin{pmatrix} 1+2\Phi & 0 & 0 & 0 \\ 0 & -1+2\phi & 0 & 0 \\ 0 & 0 & -1+2\phi & 0 \\ 0 & 0 & 0 & -1+2\phi \end{pmatrix}. \tag{16.13}$$

Here $\bar{\delta}_{jk}$ is a tensor whose components, in the rest frame of the fluid, are equal to those of the unit matrix. (It is not the same as the Kronecker tensor $\delta^j{}_k$, which has these same components in all reference frames.)

Also, the divergence of the energy momentum tensor must be replaced by the *covariant divergence*,

$$\nabla_k T^{jk} = \frac{1}{\sqrt{-g}}\partial_k(\sqrt{-g}T^{jk}) - \frac{1}{2}T^{lk}\partial^j g_{lk} \tag{16.14}$$

where $g = \det(g_{jk}) \simeq -(1-4\Phi)$ is the determinant of the metric tensor.

Keeping terms to first order in $\Phi$ we get

$$\begin{aligned}\nabla_k T^{jk} &= \frac{1}{1-2\Phi}\partial_k[(1-2\Phi)T^{jk}] - T^{lk}\bar{\delta}_{lk}\partial^j\Phi, \\ &= \partial_k T^{jk} - 2T^{jk}\partial_k\Phi - (\rho+3P)\partial^j\Phi, \\ &= \partial_k T^{jk} - 2(\rho+P)u^j\frac{d\Phi}{d\tau} - (\rho+P)\partial^j\Phi. \end{aligned} \tag{16.15}$$

Therefore,

$$u_j \nabla_k T^{jk} = u_j \partial_k T^{jk} - 3(\rho + P)\frac{d\Phi}{d\tau} = 0. \tag{16.16}$$

From this we see that the continuity equation becomes

$$\partial_k[(\rho + P)u^k] = \frac{dP}{d\tau} + 3(\rho + P)\frac{d\Phi}{d\tau} \tag{16.17}$$

and the Euler equation becomes

$$(\rho + P)\frac{du^j}{d\tau} = \partial^j P - u^j \frac{dP}{d\tau} + (\rho + P)\left(\partial^j \Phi - u^j \frac{d\Phi}{d\tau}\right) \tag{16.18}$$

The nonrelativistic counterparts are

$$\frac{\partial}{\partial t}\rho(1 - 3\Phi) = -\nabla \cdot \rho \boldsymbol{v} + 3\rho \boldsymbol{v} \cdot \nabla\Phi, \tag{16.19}$$

$$\rho \frac{d}{dt}\boldsymbol{v}(1 + \Phi) = -\nabla P - \rho \nabla \Phi. \tag{16.20}$$

## 16.4 Hydrostatic equilibrium and polytropes

An important special case is that of a static fluid in equilibrium. A star, for example. Then the time derivatives vanish and the Euler equation reduces to the *equation of hydrostatic equilibrium*

$$\nabla P = -\rho \nabla \Phi. \tag{16.21}$$

For the case of spherical symmetry this becomes

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr} = -\rho g(r) \tag{16.22}$$

where $g(r)$ is the local gravitational acceleration. This is one of the fundamental equations of stellar structure.

Taking the divergence of (16.21) and using Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{16.23}$$

we obtain

$$\nabla \cdot \frac{1}{\rho} \nabla P = -4\pi G \rho. \tag{16.24}$$

To simplify this further, suppose that the equation of state of the gas has the form

$$P = K\rho^{1+1/n}. \tag{16.25}$$

where $n$ is a constant called the *polytropic index*. For example an ideal gas at constant temperature would have $n = \infty$ and $K = T/m$. Substituting for $P$ gives an equation for $\rho$,

$$\nabla \cdot \frac{1}{\rho} \nabla \rho T = -4\pi m G \rho. \tag{16.26}$$

In the case of an *isothermal* gas, $T$ is a constant so this becomes

$$\nabla^2 \ln \rho = -\frac{4\pi G n}{K(n+1)}\rho. \tag{16.27}$$

In the case of spherical symmetry, this is the equation for an *ideal gas sphere*, which gives a good approximation for the density profiles of hot gas in clusters of galaxies.

More generally, the substitutions $\rho = \rho_c \theta^n$ and $r^2 = [(n+1)K\rho_c^{(1-n)/n}/4\pi G]\xi^2$ gives the dimensionless *Lane-Emden equation*,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0. \tag{16.28}$$

This non-linear equation can be solved numerically, typically with initial conditions $\theta(0) = \theta'(0) = 0$. The resulting solutions describe *polytropic gas spheres* or polytropes, which are a useful approximation in the theory of stellar structure.

## 16.5 Viscosity

So far we have restricted our attention to perfect fluids. In fact all fluids exhibit some degree of viscosity. Viscosity describes the transfer of momentum within the fluid, either by interactions of the fluid particles or by diffusion. To account for this we must add to the energy momentum tensor a term $\sigma^{jk}$ that involves only spatial derivatives of the velocity, and which vanishes in the case of a pure rotation (see Landau and Lifshitz, em Fluid Mechanics).

In three dimensions, the most general form that satisfies these conditions is $\partial_\alpha v_\beta + \partial_\beta v_\alpha$. We separate this into two parts, a trace-free tensor and a diagonal tensor, multiplied by coefficients $\eta$ and $\zeta$ that are independent of the velocity. Thus the three-dimensional viscous stress tensor is

$$\sigma_{\alpha\beta} = \eta(\partial_\alpha v_\beta + \partial_\beta v_\alpha - \frac{2}{3}\delta_{\alpha\beta}\partial_\gamma v^\gamma) + \zeta\delta_{\alpha\beta}\partial_\gamma v^\gamma. \tag{16.29}$$

The four-dimensional viscous stress tensor can now be inferred. To do this we note that in the rest frame, $\sigma^{jk}$ does not contribute to the energy density or overall momentum flux, so in this frame $\sigma^{00} = \sigma^{0\alpha} = 0$. Therefore, we must multiply the four-dimensional version of (16.29) by the projection operator $g_{jk} - u_j u_k$ in order to eliminate the 00 and $0\alpha$ components. This leads us to the result

$$\sigma^{jk} = -\eta[(\partial^j - u^j u^l \partial_l)u^k + (\partial^k - u^k u^l \partial_l)u^j] - (\zeta - \frac{2}{3}\eta)(g^{jk} - u^j u^k)\partial_l u^l. \tag{16.30}$$

$\eta$ and $\zeta$ are called *coefficients of viscosity*. They are functions of pressure and temperature, but in many cases may be regarded as constant within the fluid.

Including viscosity, but not gravity, the nonrelativistic form of the Euler equation becomes

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla P + \eta\nabla^2\boldsymbol{v} + (\zeta + \frac{1}{3}\eta)\nabla(\nabla\cdot\boldsymbol{v}). \tag{16.31}$$

which is called the *Navier-Stokes equation*.

## 16.6 The Jeans instability

Sir James Jeans (1902, Philosophical Transactions of the Royal Society A 199, 1) first examined the stability of a gas cloud. He showed that gravitational collapse occurs if the cloud is massive enough for gravity to overcome the internal gas pressure. The minimum stable mass is called the *Jeans mass.*

To analyze this, we start with the nonrelativistic equations for a perfect fluid coupled to gravity, derived in the previous section. In the case of a weak gravitational field, these are

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \boldsymbol{v}, \tag{16.32}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi. \tag{16.33}$$

To these we add the equation of state and Poisson's equation,

$$P = c_s^2 \rho, \tag{16.34}$$

$$\nabla^2 \Phi = 4\pi G \rho. \tag{16.35}$$

Here $c_s = \sqrt{T/m}$ is the sound speed in the gas, which depends on temperature $T$ and mean molecular mass $m$. We assume that the cloud is isothermal and has a homogeneous composition so that $c_s$ is constant. We now eliminate the variables $P$ and $\Phi$ to get

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \boldsymbol{v}, \tag{16.36}$$

$$\nabla \cdot \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = -c_s^2 \nabla \cdot \frac{1}{\rho} \nabla \rho - 4\pi G \rho. \tag{16.37}$$

The second equation is nonlinear, but in order to determine stability, we need only consider small departures from the equilibrium state. This allows us to linearize the equations. We write

$$\rho = \rho_0 + \rho_1, \tag{16.38}$$

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1, \tag{16.39}$$

where the subscript 1 denotes a small perturbation. Thus, $\rho_1 << \rho_0$. The subscript 0 denotes the equilibrium values of the parameters. We suppose that the cloud is initially at rest, so $\boldsymbol{v}_0 = 0$.

Now substitute these expressions in the equations and keep only terms up to first order in the perturbations. This gives

$$\frac{\partial}{\partial t}(\rho_0 + \rho_1) = -\nabla \cdot \rho_0 \boldsymbol{v}_1, \tag{16.40}$$

$$\nabla \cdot \frac{\partial \boldsymbol{v}_1}{\partial t} = -c_s^2 \nabla \cdot \frac{1}{\rho_0 + \rho_1} \nabla(\rho_0 + \rho_1) - 4\pi G(\rho_0 + \rho_1). \tag{16.41}$$

The equations must also be satisfied if we set all the perturbations equal to zero, since the perturbed configuration was, by assumption, a solution. Thus,

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{16.42}$$

$$0 = -c_s^2 \nabla \cdot \frac{1}{\rho_0} \nabla \rho_0 - 4\pi G \rho_0. \tag{16.43}$$

Subtracting the unperturbed equations from the perturbed equations produces the *linearized equations*,

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \boldsymbol{v}_1 - \boldsymbol{v}_1 \cdot \nabla \cdot \rho_0 \tag{16.44}$$

$$\nabla \cdot \frac{\partial \boldsymbol{v}_1}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla^2 \rho_1 + 4\pi G \rho_1. \tag{16.45}$$

We shall assume that the density distribution is initially smooth, so that the second term on the RHS of the first equation is negligible. (This is called the Jeans swindle.)

We now look for plane-wave solutions, in the form

$$\rho_1 = \bar{\rho}_1 e^{i(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})}, \tag{16.46}$$

$$\boldsymbol{v}_1 = \bar{\boldsymbol{v}}_1 e^{i(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})}, \tag{16.47}$$

where the barred symbols are constants. Substituting these in the linearized equations, we get

$$i\omega \bar{\rho}_1 = i\rho_0 \boldsymbol{k} \cdot \bar{\boldsymbol{v}}_1 \tag{16.48}$$

$$\omega \boldsymbol{k} \cdot \bar{\boldsymbol{v}}_1 = \frac{c_s^2}{\rho_0} k^2 \bar{\rho}_1 + 4\pi G \bar{\rho}_1. \tag{16.49}$$

Eliminating the velocity, we get the *dispersion equation.*

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0. \tag{16.50}$$

On small scales, $k \gg 4\pi G\rho/c_s^2$, gravity is unimportant and we can ignore the last term. The solution then corresponds to acoustic waves propagating at the sound speed $c_s$. However on larger scales the situation changes. if $k^2 < 4\pi G\rho/c_s^2$ then $\omega^2 < 0$ and the density grows exponentially with time. Thus, fluctuations having a characteristic scale $\lambda = 2\pi/k$ greater than the *Jeans length*

$$\lambda_J = c_s \sqrt{\frac{\pi}{G\rho}} = \left( \frac{\pi T}{G\rho\mu m_H} \right)^{1/2} \tag{16.51}$$

are unstable.

The quantity $1/\sqrt{G\rho}$ has units of time and is roughly equal to the gravitational collapse time. We see that the Jeans length is roughly the distance travelled by a sound wave in one collapse time. On scales larger than this there is no way for thermal pressure to stabilize the cloud before it collapses.

For a spherical cloud, the Jeans mass is

$$M_J = \frac{4\pi\rho}{3} \left( \frac{\lambda_J}{2} \right)^3 = \frac{\pi^{5/2} c_s^3}{6 G^{3/2} \rho^{1/2}}. \tag{16.52}$$

In SI units it has the value

$$M_J = 1.559\, T^{3/2} \mu^{-2} \left( \frac{n}{10^9} \right)^{-1/2} M_\odot. \tag{16.53}$$

# 17 Plasmas

Plasmas are gasses that are highly ionized. They constitute the matter of stars, the hot interstellar and intergalactic medium, magnetospheres, accretion disks and astrophysical jets.

Because the particles in a plasma are charged, they are strongly affected by electromagnetic fields. Thus, we will need to modify the equations of fluid dynamics to include electromagnetic forces. This will lead to the study of magnetohydrodynamics. But first, we examine the propagation of electromagnetic waves in a plasma.

## 17.1 Plasma frequency

When an EM wave propagates in a plasma, the oscillating electric field in the wave produces in oscillating motion of the electrons. This in turn generates an electromagnetic field that combines that of the wave, altering the propagation. Ions also contribute, but their much larger mass results in a much lower acceleration so the effect of the ions is tiny.

Recall that the Lorentz force on a single electron gives

$$m\frac{du^j}{d\tau} = -eF^{jk}u_k \tag{17.1}$$

Moving electrons constitute a current. If the electron density (in the rest frame of the fluid) is $n$, the current density is

$$j^j = -enu^j. \tag{17.2}$$

Therefore,

$$\frac{dj^j}{d\tau} = \frac{ne^2}{m}F^{jk}u_k \tag{17.3}$$

In turn, the current density generates an electromagnetic field which adds to that of the wave. This induced field is given by Maxwell's equations,

$$\partial_j F^{jk} = j^k \tag{17.4}$$

Thus

$$\frac{d}{d\tau}\partial_j F^{jk} = \frac{ne^2}{m}F^{kj}u_j = -\frac{ne^2}{m}u_j F^{jk} \tag{17.5}$$

If there is no static electric or magnetic field, $F^{jk}$ represents only the electromagnetic wave and can be written in the form (for a plane wave) $F^{jk} = \mathcal{F}^{jk}e^{i\vec{k}\cdot\vec{x}}$. Therefore

$$\left[(iu^l k_l)ik_j + \frac{ne^2}{m}u_j\right]\mathcal{F}^{jk} = 0. \tag{17.6}$$

Since $\mathcal{F}^{jk}$ represents an arbitrary wave, it follows that all components of the vector in brackets must be zero. Therefore,

$$(iu^l k_l)ik_j = -\frac{ne^2}{m}u_j. \tag{17.7}$$

Multiplying by $k^j$ and contracting, we get

$$k^2 = \frac{ne^2}{m} \tag{17.8}$$

This is the dispersion relation for the wave. In $3+1$ notation it becomes

$$\boldsymbol{k}^2 = \omega^2 - \omega_p^2, \tag{17.9}$$

where

$$\omega_p^2 = \frac{ne^2}{m} \tag{17.10}$$

is called the *plasma frequency.*

The phase velocity of the wave is given by

$$c_p = \frac{\omega}{|\boldsymbol{k}|} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}, \tag{17.11}$$

and the group velocity is

$$c_g = \frac{d\omega}{d|\boldsymbol{k}|} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}. \tag{17.12}$$

We see that the phase velocity exceeds the speed of light, but of course the group velocity does not.

It is evident that a problem occurs if $\omega < \omega_p$. The modulus of the wave vector $\boldsymbol{k}$ then becomes imaginary! This means that the wave no longer oscillates, but is exponentially attenuated. Thus low-frequency waves, having $\omega \leqslant \omega_p$ cannot propagate in a plasma.

The variation of the group velocity with frequency gives rise to *dispersion.* A pulse of radiation, produced by a pulsar for example, contains a range of frequencies. These propagate at different speeds through the interstellar medium. As a result, the pulse width increases with distance. The time required for a frequency $\omega$ to travel a distance $d$ is

$$t = \int_0^d \frac{ds}{v_g} = \int_0^d \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} ds, \tag{17.13}$$

$$\simeq \int_0^d \left(1 + \frac{\omega_p^2}{2\omega^2}\right) ds, \tag{17.14}$$

$$= d + \frac{e^2}{m\omega^2}\mathcal{D}, \tag{17.15}$$

where

$$\mathcal{D} = \int_0^d n ds, \tag{17.16}$$

is the *dispersion measure.*

## 17.2 Faraday rotation

If a magnetic field is present, the polarization of propagating waves is affected. Linearly-polarized radiation can be written as the sum of equal right and left hand circularly-polarized components,

$$\boldsymbol{E} = \mathcal{E}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon_2})e^{i\omega t}, \tag{17.17}$$

where $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_1$ are orthogonal unit vectors in the plane transverse to the propagation direction. Each component will induce a circular motion of the electrons with frequency $\omega$, around the direction of propagation, but in opposite directions.

The presence of a magnetic field component parallel to the direction of propagation changes the frequency of the orbital motion to $\omega \pm \omega_B$, where

$$\omega_B = \frac{eB}{\gamma m} \tag{17.18}$$

is the cyclotron frequency. As a result, the dispersion relation becomes

$$\boldsymbol{k}^2 = \omega(\omega \pm \omega_B) - \omega_p^2 \tag{17.19}$$

and we have a different group velocity for the RHC and LHC polarizations. Because of this, the plane of polarization of linearly-polarized light will rotate as it propagates. The change in polarization angle after propagating a distance $d$ is given by (Rybicki and Lightman),

$$\Delta\theta = \frac{e^3}{2m^2\omega^2}\int_0^d nB_{\|}ds \tag{17.20}$$

where $B_{\|}$ is the component of the magnetic field along the line of sight.

Measurements of the rotation at several frequencies can provide information about the field strength if the electron density is known (from dispersion measurements for example). However, if the field changes direction along the line of sight, this will provide only a lower limit.

## 17.3 Magnetohydrodynamics

An ionized fluid is subject to electric and magnetic forces. We can accommodate this by adding the volume density of the Lorentz four-force to the relativistic Euler equation. If viscosity can be ignored, we obtain

$$(\rho + P)\frac{du^i}{d\tau} = \partial^i P - u^i\frac{dP}{d\tau} + F^{ik}j_k, \tag{17.21}$$

where $j_j$ is the four-current density. In turn, fluid motion generates magnetic fields, $\partial_j F^{jk} = j_k$. Thus we arrive at the equation of motion

$$(\rho + P)\frac{du^i}{d\tau} = \partial^i P - u^i\frac{dP}{d\tau} + F^{ij}\partial^k F_{kj}, \tag{17.22}$$

We could have arrived at the same result by adding to the energy momentum tensor of the electromagnetic field to that of the fluid. In natural units, it is

$$T^{jk} = F^{jl} F_l{}^k + \frac{1}{4}\eta^{jk} F^{lm} F_{lm}. \tag{17.23}$$

Taking the divergence of this tensor, and projecting orthogonal to $u$, gives the same Lorentz term.

In $3+1$ notation, the relativistic Euler equation becomes

$$\gamma(\rho + P)\frac{d\gamma}{dt} = \frac{\partial P}{\partial t} - \gamma^2 \frac{dP}{dt} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{B} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t}, \tag{17.24}$$

$$\gamma(\rho + P)\frac{d\gamma \boldsymbol{v}}{dt} = -\nabla P - \boldsymbol{v}\frac{\partial P}{\partial t} + (\nabla \cdot \boldsymbol{E})\boldsymbol{E} - \boldsymbol{B} \times \nabla \times \boldsymbol{B} + \boldsymbol{B} \times \frac{\partial \boldsymbol{E}}{\partial t}, \tag{17.25}$$

Multiplying the first equation by $\boldsymbol{v}$ and subtracting it from the second, we get

$$\gamma^2(\rho + P)\frac{d\boldsymbol{v}}{dt} = -\nabla P - \gamma^2 \boldsymbol{v}\frac{\partial P}{\partial t} - \boldsymbol{B} \times \boldsymbol{J} + \rho_e \boldsymbol{E} + (\boldsymbol{E} \cdot \boldsymbol{J})\boldsymbol{v}. \tag{17.26}$$

where

$$\rho_e = \nabla \cdot \boldsymbol{E}, \tag{17.27}$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} \tag{17.28}$$

are the three-dimensional charge and current densities. Generally, the time derivative of the electric field is small compared to the curl of the magnetic field and we can just take $\boldsymbol{J} = \nabla \times \boldsymbol{B}$.

If we now assume that the motion is non-relativistic, (17.26) reduces to

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla P - \boldsymbol{B} \times \boldsymbol{J} + \rho_e \boldsymbol{E} + (\boldsymbol{E} \cdot \boldsymbol{J})\boldsymbol{v}. \tag{17.29}$$

The terms on the RHS correspond to the pressure force, the magnetic force, the electrostatic force, and a drag term due to energy dissipation by the current. Generally, the last two terms can be neglected and the equation reduces to

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla P - \boldsymbol{B} \times (\nabla \times \boldsymbol{B}), \tag{17.30}$$

The current density produced by the Lorentz force, in the non-relativistic limit, is given by

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{17.31}$$

where $\sigma$ is the *conductivity* of the plasma. In general, the conductivity is large, which means that

$$\boldsymbol{E} \simeq -\boldsymbol{v} \times \boldsymbol{B}. \tag{17.32}$$

From this we see that

$$\boldsymbol{E} \cdot \boldsymbol{B} \simeq -(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{B} = -\boldsymbol{v} \cdot (\boldsymbol{B} \times \boldsymbol{B}) = 0. \tag{17.33}$$

so the electric and magnetic fields are orthogonal. (Physically, if the conductivity is high the charges will quickly move along the magnetic field lines to cancel any parallel component of the electric field.)

Maxwell's equations provide the following additional relations,

$$\nabla \cdot \boldsymbol{B} = 0, \tag{17.34}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}). \tag{17.35}$$

It is worth noting that particle motion in the presence of a magnetic field is conservative. The Lorentz force $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$ produces an acceleration that is perpendicular to the velocity. Therefore $dE/dt = \boldsymbol{v} \cdot \boldsymbol{F} = 0$. The magnetic field does no work and, if there are no non-magnetic forces, the kinetic energy of the particle is constant.

## 17.4 Adiabatic invariants

In a system undergoing periodic motion, in which some parameter varies slowly, compared to the period, there is a conserved quantity $\mathcal{I}$ called an adiabatic invariant (see for example, Landau and Lifshitz, *Mechanics*, §49). If $p$ and $q$ are conjugate variables, the adiabatic invariant can be expressed as an integral along the classical trajectory of the system in phase space,

$$\mathcal{I} = \oint \boldsymbol{p} \cdot d\boldsymbol{q}. \tag{17.36}$$

Let's apply this to the orbital motion of a charge $e$ in a plane perpendicular to the magnetic field. The generalized momentum is given by

$$\boldsymbol{p} = \gamma m \boldsymbol{v}_\perp + q\boldsymbol{A} \tag{17.37}$$

where $\boldsymbol{A}$ is the vector potential. The generalized coordinate is the position vector $\boldsymbol{r}$ of the charge. Thus

$$\mathcal{I} = 2\pi r \gamma m v_\perp + q \oint \boldsymbol{A} \cdot d\boldsymbol{r}. \tag{17.38}$$

The line integral can be converted to a surface integral using Stokes theorem,

$$\begin{aligned} \mathcal{I} &= 2\pi r \gamma m v_\perp - q \int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S}, \\ &= 2\pi r \gamma m v_\perp - q \int \boldsymbol{B} \cdot d\boldsymbol{S}, \\ &= 2\pi r \gamma m v_\perp - q\pi r^2 B. \end{aligned} \tag{17.39}$$

Here $r$ is the radius of the orbit, which is given by

$$r = \frac{v_\perp}{\omega_B} = \frac{\gamma m v_\perp}{qB} \tag{17.40}$$

Substituting this in the previous equation gives

$$\mathcal{I} = \pi \frac{\gamma^2 m^2 v_\perp^2}{qB} = q\pi r^2 B \tag{17.41}$$

Therefore, for nonrelativistic motion, the magnetic moment

$$\mu = \frac{m v_\perp^2}{2B} \tag{17.42}$$

is a an adiabatic invariant. For relativistic motion, multiply this by $\gamma^2$. In all cases, the magnetic flux $\phi = \pi r^2 B$, enclosed by the orbit, is invariant.

## 17.5 Magnetospheres

An example of a plasma coupled to a magnetic field is a planetary magnetosphere. The magnetic field is produced by currents circulating in a liquid core of the planet and has a dipole structure (Fig 17.1). Charged particles from the solar (or stellar) wind penetrate the magnetosphere and are captured into orbits that spiral around magnetic field lines.

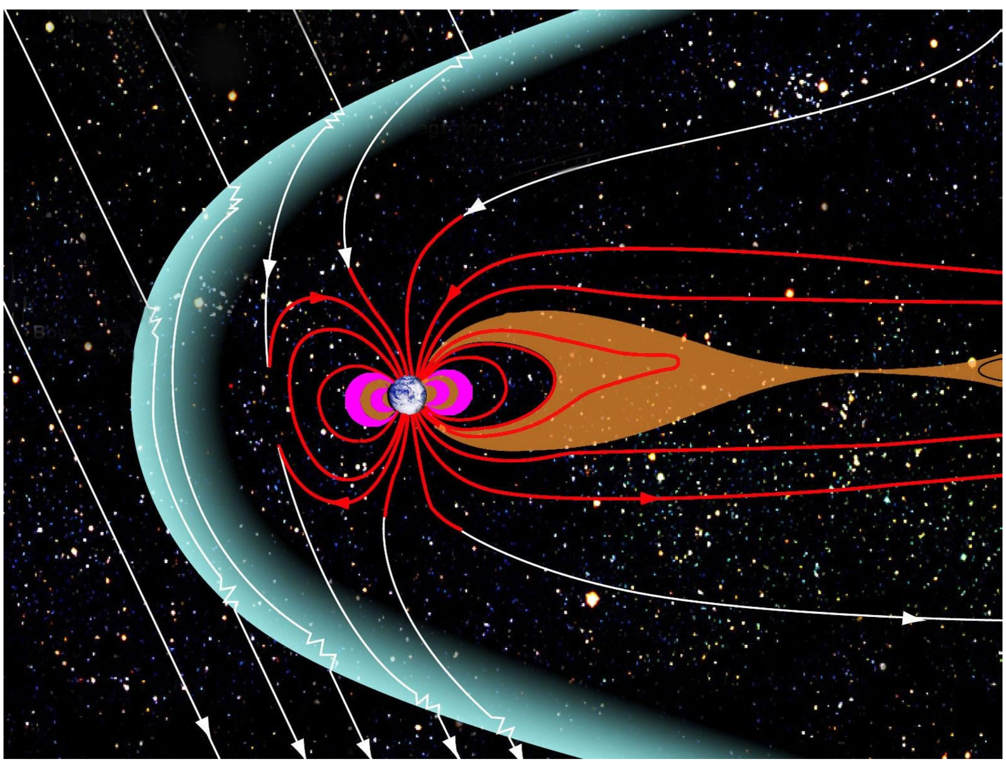


Figure 17.1: Sketch of the Earth's magnetosphere. Lines show the magnetic field and the blue surface represents the magnetopause, the boundary between the interplanetary and terrestrial magnetic field. The purple region shows the Van-Allen belts, a region of high density of trapped charged particles. (credit: NASA)

Where the energy density of the magnetic field $B^2/2$ exceeds that of the particles $\rho v^2/2$, the magnetic field is not much affected by the particles. The particles move in helical orbits along the magnetic field lines. Where the energy density of the particles exceeds that of the magnetic field, the magnetic field lines are dragged by the flow of the plasma.

In the inner region of the magnetosphere, where the magnetic field strength is high, charged particles become trapped. As they move along the field lines the field strength increases as they approach the planet. Since the magnetic moment of the particle's orbital motion is an adiabatic invariant, the particle's transverse kinetic energy $mv_\perp^2/2$ must increase in proportion to the field strength. The total energy of the particle remains constant, so the velocity component $v_\parallel$, parallel to the magnetic field must decrease. At some point the parallel velocity reaches zero and then reverses direction. The particle then spirals backward, away from the planet. This is called a *magnetic mirror*.

Particles trapped in the magnetosphere travel forwards and backwards along the magnetic field, between the mirror points. At the point of closest approach to the planet, they may enter the upper atmosphere. Collisions between energetic charged particles and atoms or molecules in the atmosphere excite the atoms which then radiate producing an aurora. The terrestrial aurora consists primarily of emission lines of oxygen and hydrogen which arise in the upper mesosphere and lower thermosphere, some 100 km or more above the Earth's surface.

## 17.6 Fermi acceleration

Enrico Fermi first proposed a mechanism for accelerating particles by interaction with magnetic fields. We have seen that converging field lines, corresponding to increasing magnetic field strength, can reflect charged particles. If the field configuration is moving, the reflected particles will gain or loose energy, just as if they were being reflecting by a moving surface.

A collision a particle and an approaching mirror will increase the particle energy, while a particle reflected by a receding mirror will loose energy. However, Fermi realized that collisions with an approaching mirror happen *more frequently* than collisions with a receding mirror. The rate with which a mirror collides with particles is given by the usual formula $n\sigma v$, where $\sigma$ is the number density of targets (particles in this case), $\sigma$ is the collision cross-section and $v$ is the *relative* velocity. This is obviously higher for approaching collisions that for receding collisions.

Fermi acceleration is believed to be a principal mechanism for the acceleration of cosmic ray particle by interstellar magnetic fields and shock waves.

## 17.7 Hydromagnetic waves

To investigate waves in a plasma, we consider small perturbations $\rho_1$, $P_1$, $\boldsymbol{B}_1$, $\boldsymbol{v}$ from an equilibrium state in which the fluid is at rest in a uniform magnetic field $B$. Substituting $\rho = \rho_0 + \rho_1$, etc, into the non-relativisitic equations gives the linearized equations

$$\nabla \cdot \boldsymbol{B}_1 = 0, \tag{17.43}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{17.44}$$

$$\frac{\partial \rho_1}{\partial t} = -\rho \nabla \cdot \boldsymbol{v}, \tag{17.45}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = -c_s^2 \nabla \rho_1 - \boldsymbol{B} \times (\nabla \times \boldsymbol{B}_1), \tag{17.46}$$

where we have dropped the unnecessary subscript 0.

We now look for plane-wave solutions proportional to $\exp[i(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})]$. Substitution in the linearized equations then gives a system of algebraic equations,

$$\boldsymbol{k} \cdot \boldsymbol{B_1} = 0, \tag{17.47}$$

$$\omega \boldsymbol{B_1} = -\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{17.48}$$

$$\omega \rho_1 = \rho \boldsymbol{k} \cdot \boldsymbol{v}, \tag{17.49}$$

$$\rho \omega \boldsymbol{v} = +c_s^2 \rho_1 \boldsymbol{k} + \boldsymbol{B} \times (\boldsymbol{k} \times \boldsymbol{B}_1), \tag{17.50}$$

The first equation is satisfied automatically by the second, which shows that $\boldsymbol{B}_1$ is perpendicular to $\boldsymbol{k}$.

Solve the third equation for $\rho_1$ and substitute this in the fourth. The equations become

$$\begin{aligned} \omega \boldsymbol{B_1} &= -\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B}), \\ &= -(\boldsymbol{k} \cdot \boldsymbol{B}) \boldsymbol{v} + (\boldsymbol{k} \cdot \boldsymbol{v}) \boldsymbol{B} \end{aligned} \tag{17.51}$$

$$\begin{aligned} \rho \omega \boldsymbol{v} &= \frac{c_s^2}{\omega} \rho (\boldsymbol{k} \cdot \boldsymbol{v}) \boldsymbol{k} + \boldsymbol{B} \times (\boldsymbol{k} \times \boldsymbol{B_1}), \\ &= \left[ \frac{c_s^2}{\omega} \rho (\boldsymbol{k} \cdot \boldsymbol{v}) + (\boldsymbol{B} \cdot \boldsymbol{B}_1) \right] \boldsymbol{k} - (\boldsymbol{B} \cdot \boldsymbol{k}) \boldsymbol{B}_1 \end{aligned} \tag{17.52}$$

to simplify this, set up a Cartesian coordinate system with $\boldsymbol{k}$ lying along the $x$ axis and $\boldsymbol{B}$ lying in the $x - y$ plane. The $z$ component of our equations is

$$\omega B_{1z} = -(\boldsymbol{k} \cdot \boldsymbol{B}) v_z, \tag{17.53}$$

$$\rho \omega v_z = (\boldsymbol{k} \cdot \boldsymbol{B}) B_{1z}, \tag{17.54}$$

where the symbols $\|$ and $\perp$ denote components parallel and perpendicular to $\boldsymbol{k}$. Combining these two equations, we find the dispersion relation

$$\omega = \frac{\boldsymbol{k} \cdot \boldsymbol{B}}{\sqrt{\rho}}. \tag{17.55}$$

These correspond to waves propagating with speed

$$v_A = \frac{B_\parallel}{\sqrt{\rho}} \tag{17.56}$$

which are called *Alfvén waves* (H. Alfvén, 1942). The physical propagation velocity (group velocity) is

$$\boldsymbol{v} = \frac{\partial \omega}{\partial \boldsymbol{k}} = \frac{\boldsymbol{B}}{\sqrt{\rho}}, \tag{17.57}$$

which shows that the wave energy propagates in the direction of the magnetic field $\boldsymbol{B}$, not the direction $\boldsymbol{k}$.

Let's now look at the $x$ and $y$ components of the equations,

$$\omega B_{1y} = -kB_x v_y + kv_x B_y, \tag{17.58}$$

$$\rho\omega v_y = -kB_x B_{1y}, \tag{17.59}$$

$$\rho\omega v_x = \frac{c_s^2}{\omega}\rho k^2 v_x + kB_y B_{1y}. \tag{17.60}$$

Eliminating the variables $v_x$, $v_y$ and $B_{1y}$ gives the dispersion relation

$$(\omega^2 - c_s^2 k^2)(\rho\omega^2 - k^2 B_x^2) = \omega^2 k^2 B_y^2. \tag{17.61}$$

This is a quadratic equation for $\omega^2$, which has the solutions

$$\omega^2 = \frac{k^2}{2}\left\{ c_s^2 + \frac{B^2}{\rho} \pm \left[ \left( c_s^2 + \frac{B^2}{\rho} \right)^2 - 4c_s^2 \frac{B_x^2}{\rho} \right]^{1/2} \right\}. \tag{17.62}$$

The two solutions correspond to *fast* and *slow magnetosonic waves.* Note that as $B \leftarrow 0$, we have $v_y \rightarrow 0$ and $\omega^2 \rightarrow c_s^2 k^2$ and the fast magnetosonic waves reduce to ordinary acoustic waves. The presence of a magnetic field induces transverse oscillations in the magnetic field and fluid velocity. In the same limiting case, the velocity of the slow magnetosonic waves approaches the Alfvén velocity. These become Alfvén waves, but with a different polarization.

Alfvén waves can be though of as transverse waves propagating along the magnetic field lines, much like a vibrating string. The restoring force is provided by the magnetic tension $B^2$ and the inertia by the mass density $\rho$.

# 18 Shock waves

Shock waves occur in many astrophysical environments, wherever supersonic gas flows occur. For example, a *bow shock* forms when the solar wind encounters a planetary magnetosphere (Figure 17.1). Exploding stars create shock waves when the supersonic ejecta encounter the interstellar medium, and also within the star itself when the core collapses. The external shock compresses and heats the ISM and may trigger star formation. Encounters between galaxies create shock waves when gas clouds collide. This is the likely source of the hot X-ray plasma found in early-type galaxies and clusters.

This section provides a brief introduction to shock waves. We shall assume a perfect gas and neglect gravity and magnetic fields. A treatment of MHD shock waves can be found in Somov 2006, *Plasma Astrophysics, Part I.*

Shock waves provide an efficient mechanism for converting kinetic energy of fluid motion into heat. The flow rapidly decelerates when passing through a shock wave, and the effect on the gas depends on conditions in the shock. It is instructive to consider to extreme cases. In an *adiabatic shock*, there is no energy loss by radiation, so energy (more precisely entropy) is conserved. In an *isothermal shock*, the released energy is efficiently radiated away and the temperature is unchanged. Real shock waves lie between these extremes.

It will be useful to introduce several thermodynamic quantities. We define $\varepsilon$ as the internal energy per unit mass. This is the energy in internal degrees of freedom, not related to the motion of the gas. It is related to the *specific enthalpy* $w$ by the thermodynamic relation

$$w = \varepsilon + PV, \tag{18.1}$$

where $V = 1/\rho$ is the volume per unit mass. The enthalpy consists of the internal energy, per unit mass, of the fluid, plus the work done against pressure forces to open up a volume $V$ for the fluid. In non-relativistic adiabatic flow of a compressible fluid, the quantity

$$w + \frac{1}{2}v^2 \tag{18.2}$$

is conserved. This is *Bernoulli's theorem*, and can be obtained by integrating Euler's equation.

In terms of the internal energy, the equation of state for a perfect gas can be written as

$$P = \rho(\gamma - 1)\varepsilon, \tag{18.3}$$

where $\gamma$ is a constant called the *adiabatic index*. For a non-relativistic monatomic gas, $\gamma = 5/3$. For a relativistic gas, $\gamma = 4/3$.

Now define the *specific entropy* $s$ to be the entropy per unit mass. The *first law of thermodynamics* takes the form

$$\begin{aligned} Tds &= d\varepsilon + PdV, \\ &= d\varepsilon - \frac{P}{\rho^2}d\rho. \qquad (18.4) \\ & \qquad (18.5) \end{aligned}$$

Therefore,

$$(\gamma - 1)Tds = d\left(\frac{P}{\rho}\right) - (\gamma - 1)\frac{P}{\rho^2}d\rho,$$
$$= \frac{dP}{\rho} - \gamma\frac{P}{\rho^2}d\rho \quad (18.6)$$

For an adiabatic change, $ds = 0$, thus

$$dP = \gamma\frac{P}{\rho}d\rho, \quad (18.7)$$

and we obtain the *adiabatic law*

$$P \propto \rho^\gamma. \quad (18.8)$$

From this we see that

$$c_s^2 = \frac{dP}{d\rho} = \gamma\frac{P}{\rho} = \gamma PV. \quad (18.9)$$

We approximate the shock as a thin surface of discontinuity, and consider what quantities must be conserved as the gas passes through it. In the rest frame of the shock, gas flows into the shock from one side and flows out on the other. We shall assume that the velocities, in this frame, are nonrelativistic. Let $v_1$ and $v_2$ be the components of the velocity perpendicular to the shock on the upstream and downstream sides (called the front and the back of the shock), respectively. Similarly let $\rho_0$ and $\rho_1$ be the density immediately upstream and downstream of the shock, etc. Conservation of mass requires that the mass flux $j = \rho v$ be the same on both sides of the shock. Similarly, the flux of momentum must also be conserved. Thus flux is $P + (\rho v)v$. Finally, for an adiabatic shock, the energy flux $\rho v(w + v^2/2)$ must be continuous. This leads us to the equations

$$\rho_1 v_1 = \rho_2 v_2 \equiv j, \quad (18.10)$$
$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2, \quad (18.11)$$
$$w_1 + \frac{v_1^2}{2} = w_2 + \frac{v_2^2}{2}. \quad (18.12)$$

These are known as the *Rankine-Hugoniot relations*, named after two engineers who first worked on this problem. They are sometimes called the *jump conditions*.

Substituting $V = 1/\rho$ and solving (18.10) for $v_1$ and $v_2$,

$$v_1 = jV_1, \quad (18.13)$$
$$v_2 = jV_2. \quad (18.14)$$

Substituting these in (18.11) we find

$$j^2 = \frac{P_2 - P_1}{V_1 - V_2}. \quad (18.15)$$

From this, we see that the speed $v_1$, at which a shock propagates through a medium, depends on the pressure and density differences across the shock. From (18.13) and (18.14) we find the velocity change across the shock,

$$v_1 - v_2 = \sqrt{(P_2 - P_1)(V_1 - V_2)}. \quad (18.16)$$

Eqn. (18.12) can be written as

$$w_1 + \frac{1}{2}j^2 V_1^2 = w_2 + \frac{1}{2}j^2 V_2^2. \tag{18.17}$$

which leads to

$$\begin{aligned} w_1 - w_2 &= \frac{1}{2}j^2(V_1^2 - V_2^2), \\ &= \frac{1}{2}(P_1 - P_2)(V_1 + V_2). \end{aligned} \tag{18.18}$$

This can also be written in terms of the internal energy,

$$\varepsilon_1 - \varepsilon_2 + \frac{1}{2}(P_1 + P_2)(V_1 - V_2) = 0 \tag{18.19}$$

These equations relate the final pressure and density to the initial pressure and density.

For an adiabatic shock, we have (from 18.1, 18.3 and 18.9),

$$\varepsilon = \frac{PV}{\gamma - 1} = \frac{c_s^2}{\gamma(\gamma - 1)}, \tag{18.20}$$

$$w = \gamma\varepsilon = \frac{c_s^2}{\gamma - 1}. \tag{18.21}$$

Substituting these in (18.19), we obtain

$$2P_1V_1 - 2P_2V_2 + (\gamma - 1)(P_1V_1 - P_1V_2 + P_2V_1 - P_2V_2) = 0. \tag{18.22}$$

Solving for $V_1/V_2$ we find

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{\gamma - 1)P_1 + (\gamma + 1)P_2}{(\gamma + 1)P_1 + (\gamma - 1)P_2}. \tag{18.23}$$

One can express the ratios of density, pressure and temperature across the shock in terms of the mach number

$$M = \frac{v_1}{c_{s1}}. \tag{18.24}$$

The result is

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma + 1)M^2}{(\gamma - 1)M^2 + 2}, \tag{18.25}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1}, \tag{18.26}$$

$$\frac{T_2}{T_1} = \frac{[2\gamma M^2 - (\gamma - 1)][(\gamma - 1)M^2 + 2]}{(\gamma + 1)^2 M}. \tag{18.27}$$

From this we see that the maximum possible value of the density ratio $\rho_2/\rho_1$ is $(\gamma + 1)/(\gamma - 1)$. For $\gamma = 5/3$ this ratio is 4. Thus, in an adiabatic shock, the gas is compressed by at most a factor of four after passing through the shock. This limit can be exceeded if the gas radiates significant energy, allowing it to cool. The adiabatic assumption then fails.