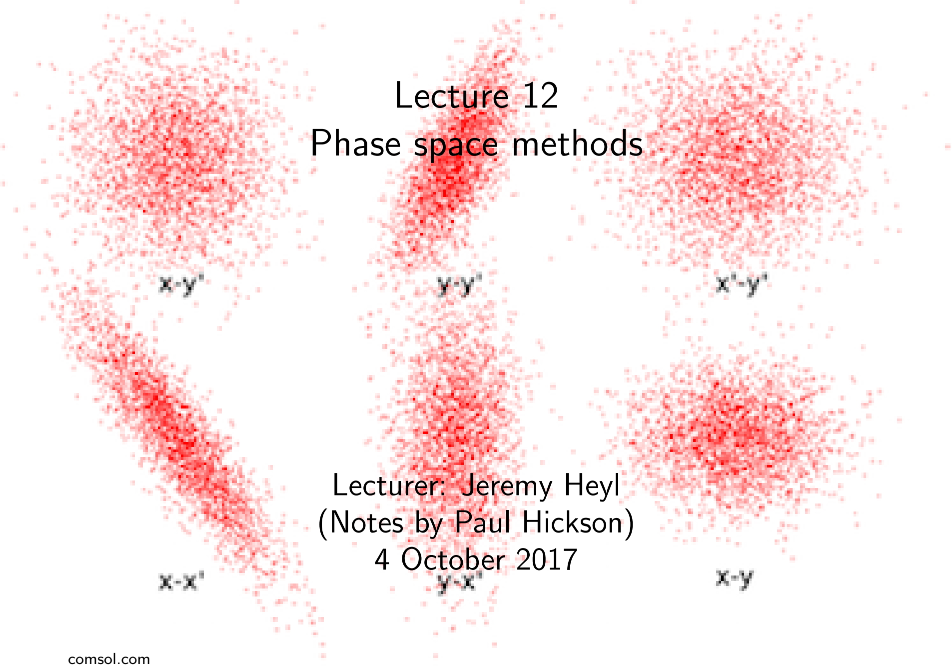
Lecture 12
Phase space methods



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Phase space distribution function

We could describe a stellar system by specifying the position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ of every star. The future evolution could then be calculated, at least in principle, by applying Newton's laws.

However, we often don't need that much information, instead a *statistical* description will suffice.

The **distribution function** or **phase space density** $f(\boldsymbol{x}, \boldsymbol{v}, t)$ describes the density of stars in six-dimensional **phase space**. Phase space has three spatial dimensions x, y, z and three velocity dimensions v_x, v_y, v_z.

The number of stars within a cube of sides Δx, Δy and Δz, centred at position $\boldsymbol{x}$ with velocity components in the range v_x to $v_x + \Delta v_x$, v_y to $v_y + \Delta v_y$ and v_z to $v_z + \Delta v_z$ is

$$\Delta N = f(\boldsymbol{x}, \boldsymbol{v}, t) \Delta x \Delta y \Delta z \Delta v_x \Delta v_y \Delta v_z.$$

Phase space distribution function

In the limit as these volumes go to zero, we can regard $f(\boldsymbol{x}, \boldsymbol{v}, t)$ as the *probability* of finding a star in the six-dimensional volume element d^3xd^3v,

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = \frac{dN}{d^3xd^3v}.$$

The number density of stars can be found by integrating the distribution function over all possible velocities,

$$n(\boldsymbol{x}, t) = \int f(\boldsymbol{x}, \boldsymbol{v}, t)d^3v.$$

The mean velocity of the stars at position $\boldsymbol{x}$ can be found by multiplying the velocity by the distribution function (probability of this velocity) and integrating,

$$\langle \boldsymbol{v}(\boldsymbol{x}, t) \rangle = \frac{1}{n(\boldsymbol{x}, t)} \int \boldsymbol{v} f(\boldsymbol{x}, \boldsymbol{v}, t)d^3v.$$

Continuity equation in three dimensions

Consider a small box of sides Δx, Δy, Δz. At time t, the number of stars in the box is $n(x,t)\Delta x\Delta y\Delta z$. After time Δt, the number of stars in the box has changed because some stars will have entered the box and some will have left,

$$\begin{aligned}\Delta N &= [n(\boldsymbol{x},t) - n(\boldsymbol{x},t+\Delta t)]\Delta x\Delta y\Delta z \\ &= v_x(x,y,z,t)n(x,y,z,t)\Delta y\Delta z\Delta t \\ &\quad - v_x(x+\Delta x,y,z,t)n(x+\Delta x,y,z,t)\Delta y\Delta z\Delta t \\ &\quad + \text{ similar terms for the y and z directions.}\end{aligned}$$

Dividing this by $\Delta x\Delta y\Delta z\Delta t$ and taking the limit as the intervals go to zero, we get the **continuity equation**

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x}(nv_x) - \frac{\partial}{\partial y}(nv_y) - \frac{\partial}{\partial z}(nv_z) = -\boldsymbol{\nabla}\cdot(n\boldsymbol{v}).$$

One-dimensional flow of stars

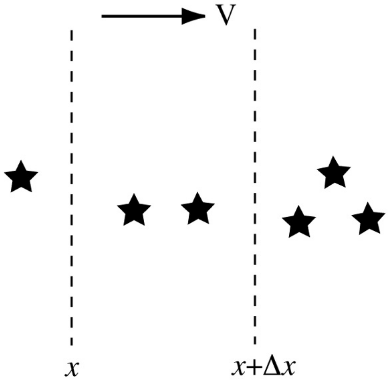

Fig 3.12 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Continuity equation in six dimensions

In the same manner, we can consider the flow of stars in phase space. This gives a six-dimensional continuity equation for the phase space density.

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla} \cdot (f\boldsymbol{v}) + \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot (f\dot{\boldsymbol{v}}) = 0$$

Here the symbol $\boldsymbol{\nabla}_{\boldsymbol{v}}$ denotes the divergence operator in *velocity space*

$$\boldsymbol{\nabla}_{\boldsymbol{v}} = \left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right)$$

Note that v_x, v_y, and v_z are treated as independent *variables*, on par with x, y and z. They are not *functions* of x, y, or z.

Flow of stars in phase space

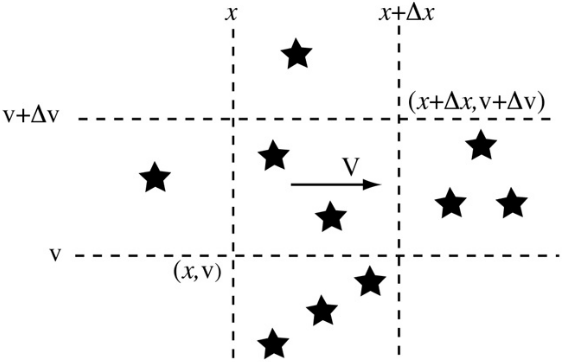

Fig 3.13 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Collisionless Boltzman equation

This six-dimensional continuity equation can be simplified by recalling that in phase space, the velocities are independent variables, not functions of x, y, z so $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$. Thus,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f + \boldsymbol{\nabla}_{\boldsymbol{v}} \cdot (f \dot{\boldsymbol{v}}) = 0$$

The acceleration $\dot{\boldsymbol{v}}$ is given by $-\boldsymbol{\nabla}\Phi$. And, Φ is a function only of position and time, It does not depend on the velocities of the stars. Therefore,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f - \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f = 0$$

This is the **collisionless Boltzmann equation** (CBE). It describes the evolution of the phase space density.

The Jeans equations

Often it is simpler to work with moments of the CBE. Integrating it over velocity, and requiring that $f \to 0$ as $v \to \infty$, gives the first Jeans equation,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n \langle \boldsymbol{v} \rangle) = 0.$$

This is the statistical equivalent of the continuity equation of fluid dynamics.

Multiplying the CBE by $\boldsymbol{v}$ and integrating over velocity gives the second Jeans equation, the equivalent of the Euler equation of fluid dynamics,

$$\frac{\partial \langle \boldsymbol{v} \rangle}{\partial t} + (\langle \boldsymbol{v} \rangle \cdot \boldsymbol{\nabla}) \langle \boldsymbol{v} \rangle = -\boldsymbol{\nabla}\Phi - \frac{1}{n}\boldsymbol{\nabla}(n\sigma^2).$$

where $\sigma(\boldsymbol{x}, t)$ is the velocity dispersion. This last term is the equivalent of a pressure gradient force.

Integrals of motion

An integral of motion is any function of the phase space coordinates $\boldsymbol{x}$ and $\boldsymbol{v}$ that is constant along the orbit of a star.

An example the energy per unit mass $E = v^2/2 + \Phi(\boldsymbol{x})$ of a stationary system (i.e. a time-independent potential).

For a spherically-symmetric potential, the angular momentum per unit mass $\boldsymbol{L}$ is an integral of motion. For an axisymmetric system, L_z is an integral of motion.

Because they are constant along the orbit, Integrals of motion $\mathcal{I}$ satisfy the equation

$$\frac{d}{dt}\mathcal{I}(\boldsymbol{x}, \boldsymbol{v}) \equiv \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla}\mathcal{I} + \dot{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_v\mathcal{I} = 0$$

Compare this to the CBE, which can be written in the form

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} f + \dot{\boldsymbol{v}} \cdot \boldsymbol{\nabla}_v f = 0$$

Jeans theorem

Comparing the two equations on the preceding slide, we see that if the distribution function f is not an explicit function of time, it remains constant along the orbits of stars.

In other words, as a star moves in its orbit, the phase space density of stars around it remains constant.

This leads us to an important theorem, due to Jeans:

Any steady-state solution of the CBE can be written as a function only of integrals of the motion, and any function of the integrals of motion is a steady-state solution of the CBE.

As an example of this, the **isothermal sphere** has the distribution function

$$f(E) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \exp\{-[v^2 + 2\Phi(r)]/2\sigma^2\}$$

Isothermal sphere

The distribution function for the isothermal sphere

$$f(E) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \exp\{-[v^2 + 2\Phi(r)]/2\sigma^2\}$$

has the form $f(E) \propto \exp(-mE/kT)$ where $T = \sigma^2/k$ is the **kinetic temperature** of the system. It is the temperature of a gas in which the atoms would have an RMS velocity equal to σ.

Integrating this gives the density,

$$\rho(r) = mn(r) = 4\pi m \int_0^\infty f(v) v^2 dv = mn_0 \exp[-\Phi(r)/\sigma^2]$$

Poisson's equation then gives us a differential equation for $\Phi(r)$,

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d\Phi(r)}{dr}\right] = 4\pi G\rho = 4\pi G m n_0 \exp[-\Phi(r)/\sigma^2].$$

This equation is nonlinear, but can be solved numerically.

King models

The isothermal distribution has infinite mass, so cannot describe real star clusters. Ivan King proposed a modified distribution,

$$f(E) = \begin{cases} \frac{n_0}{(2\pi\sigma^2)^{3/2}} \left\{\exp[-(E - \Phi_0)/2\sigma^2] - 1\right\} & E < \Phi_0 \\ 0 & E \geqslant \Phi_0 \end{cases}$$

The -1 reduces the number of stars with high kinetic energy. The resulting density drops to zero at a finite radius, mimicking the effect of tidal truncation. (Note: Eqn 3.107 in the text book is incorrect.)

These models, which are computed numerically, match the distribution of stars in globular clusters and many elliptical galaxies quite well.