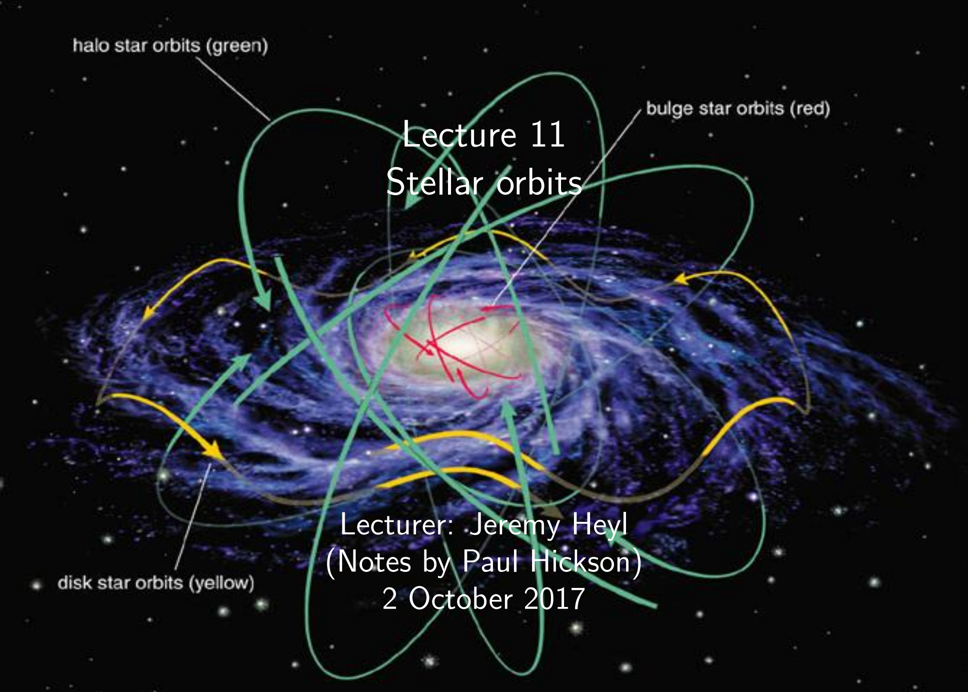
Lecture 11
Stellar orbits

Lecturer: Jeremy Heyl
(Notes by Paul Hickson)
2 October 2017



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Spherically-symmetric systems

If the gravitational potential $\Phi(r, \theta, \phi)$ is radially symmetric, we have already seen that the angular momentum of any stellar orbit $\boldsymbol{L}$ is conserved.

SInce

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{v},$$

it follows that the orbit is confined to a plane, perpendicular to $\boldsymbol{L}$.

So the star moves in a plane, but not in an ellipse. Because the mass is distributed, the gravitational force does not increase as rapidly as r decreases, compared to a point mass.

As a result, the orbit generally does not close, but traces out a *rosette* pattern. As it orbits the galaxy, the star oscillates in radius between an inner *turning point* r_{min} and an outer one r_{max}.

Rosette orbit

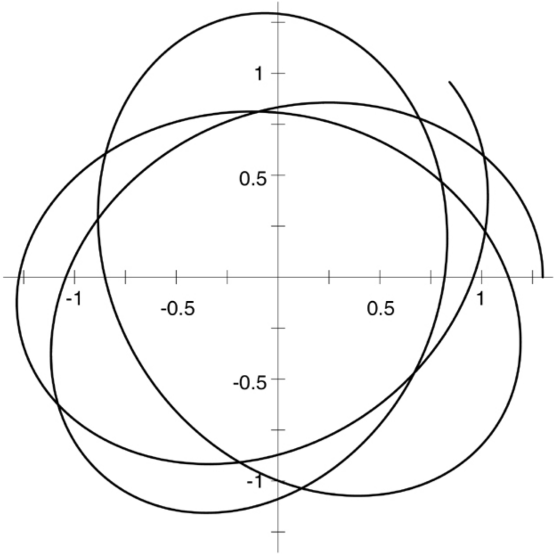

Fig 3.10 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Axisymmetric galaxies

If the gravitational potential $\Phi(R, \phi, z)$ is symmetric about the z axis, then there can be no torque about that axis. The component of angular momentum about this axis is therefore conserved,

$$L_z = R^2\dot{\phi} = constant.$$

Here, the dot indicates a derivative with respect to time.

The radial component of the acceleration of a star is

$$\ddot{R} = R\dot{\phi}^2 - \frac{\partial \Phi}{\partial R}.$$

The first term on the RHS is the centripetal acceleration.

This equation can be written as

$$\ddot{R} = -\frac{\partial \Phi_{\text{eff}}}{\partial R}, \quad \Phi_{\text{eff}} = \Phi(R, z) - \frac{1}{2}R^2\dot{\phi}^2 = \Phi(R, z) - \frac{L_z^2}{2R^2}.$$

where Φ_{eff} is the **effective potential**.

Axisymmetric galaxies

Multiply this equation by $\dot{R}$ and integrate,

$$\dot{R}\ddot{R} = \frac{1}{2}\frac{d}{dt}\dot{R}^2 = -\frac{dR}{dt}\frac{\partial\Phi_{\mathrm{eff}}}{\partial R} = \frac{d\Phi_{\mathrm{eff}}}{dt}.$$

Therefore,

$$\frac{1}{2}\dot{R}^2 + \Phi_{\mathrm{eff}} = constant.$$

The effective potential generally has the form shown on the next slide. It rises sharply as $R \to 0$ because of the centrifugal term $L_z^2/2R^2$. This prevents the star from approaching the centre to closely. Instead, it is confined to orbit within an inner and outer radial turning point, where $\dot{R}$ falls to zero.

The minimum of the effective potential occurs at $R = R_{\mathrm{g}}$, called the **guiding radius**. The star moves back and forth about this point as it circles the galaxy.

Effective potential

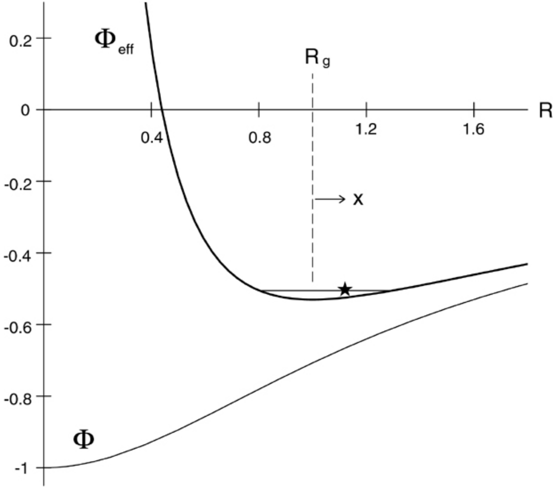

Fig 3.8 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Radial motion

If the radial motion is not too large, we can expand the effective potential in a Taylor series about the point R_{g},

$$\Phi_{\text{eff}}(R) = \Phi_{\text{eff}}(R_{\mathrm{g}}) + \frac{x^2}{2}\left[\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\right]_{R=R_{\mathrm{g}}} + \cdots \simeq L_{\text{eff}}(R_{\mathrm{g}})$$

where $x = R - R_g$. The radial equation of motion now becomes

$$\ddot{x} = -x\left[\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\right]_{R=R_{\mathrm{g}}} = -\kappa^2(R_{\mathrm{g}})x.$$

where κ is a constant called the **epicyclic frequency**. If $\kappa^2 > 0$, this is the equation of a harmonic oscillator. Its solution is

$$x = X\cos(\kappa t + \psi),$$

where X and ψ are constants. So the radial motion of the star is sinusoidal about the guiding radius.

Azimuthal motion

A star located at $R_{\rm g}$, with no radial motion, would move in a circular orbit at the circular velocity.

However, a star that is oscillating in radius must also oscillate in azimuth ϕ. This is required to conserve angular momentum L_z,

$$L_z = R^2\dot{\phi} = (R_g + x)^2\dot{\phi} \simeq (R_g^2 + 2R_g x)\dot{\phi}.$$

so

$$\dot{\phi} = \frac{L_z}{R_g^2(1 + 2x/R_g)} \simeq \Omega_g(1 - 2x/R_g),$$

where $\Omega_{\rm g} = L_z/R_{\rm g}^2$ is the angular velocity of a circular orbit at $R = R_{\rm g}$.

Substituting our solution for x and integrating, we find

$$\phi = \Omega_g t - Y\sin(\kappa t + \psi)/R_{\rm g} + \phi_0.$$

where $Y = 2\Omega_g X/\kappa$ and ϕ_0 are constants.

Epicycles

The combination of motion in x and $y = Y \sin(\kappa t + \psi)$ is illustrated in the figure.

The star moves in a **retrograde** elliptical path or *epicycle* around the guiding centre, while the guiding centre moves with constant speed in a circular path around the centre of the galaxy.

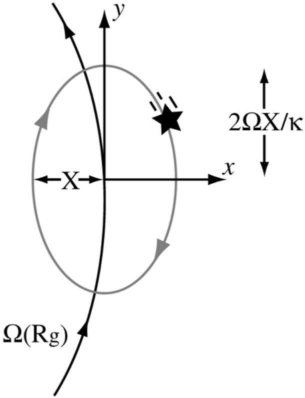

Fig 3.9 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

Rosette orbits

In general, the epicyclic frequency κ is not an exact multiple of the orbital frequency Ω_g. As a result, the orbit does not close. Instead it resembles the figure on Slide 3. The path is called a **rosette**. κ is related to the Oort constant B, as is proven in Eqn. 3.71 in Sparke and Gallagher,

$$\kappa^2(R_g) = -4B\Omega_g.$$

In the solar neighbourhood, $B < 0$, so $\kappa^2 > 0$ and the orbit is stable.

This is not true close to the event horizon of a black hole. If $R < 3r_S$, where $r_S = 2GM/c^2$ is the Schwarzschild radius, one finds that $\kappa^2 < 0$ and the orbit decays exponentially.

Stellar velocities

It is now easy to compute the velocities in the R and ϕ directions, with the overall rotation removed, by differentiating the expressions for x and y,

$$v_x = \dot{x} = X\kappa\cos(\kappa t + \psi)$$
$$v_y = \dot{y} = -Y\kappa\sin(\kappa t + \psi)$$

If we now take the time average of the squares of these velocities we find

$$\left\langle v_x^2 \right\rangle = X^2\kappa^2/2$$
$$\left\langle v_y^2 \right\rangle = Y^2\kappa^2/2$$

Looking at the definitions of X and Y, we see that

$$\left\langle v_y^2 \right\rangle = \frac{\kappa^2}{4\Omega_g^2}\left\langle v_x^2 \right\rangle.$$

Stellar velocities in the solar neighbourhood

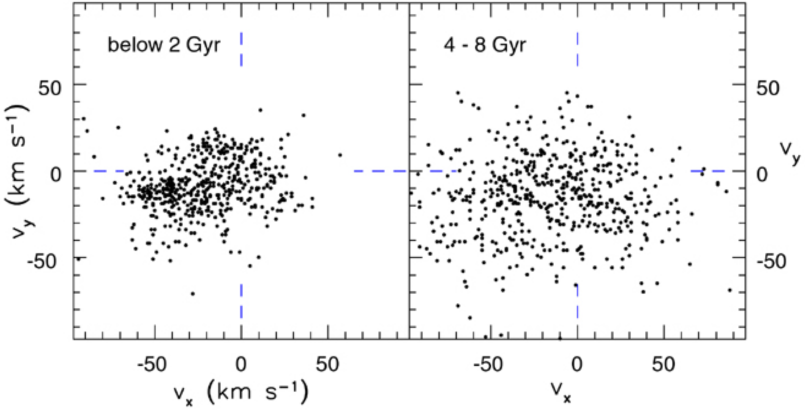

Fig 3.11 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

One can show that $\langle v_y^2 \rangle / \langle v_x^2 \rangle = -B/(A-B) \sim 2$ for a flat rotation curve. This is consistent with observations.

Motion in the z direction

In the z direction, the potential is symmetric about $z = 0$. Because there is a component of force F_z pulling a star towards the plane, the potential has a minimum at $z = 0$.

To find an approximate equation of motion for the vertical position z, we can expand the potential in a Taylor series about $z = 0$. The linear term is zero, by symmetry, so we have

$$\Phi(R, z) = \Phi(R, 0) + \frac{z^2}{2}\left[\frac{\partial^2\Phi(R, z)}{\partial^2 z}\right]_{z=0} \equiv \phi(R, 0) + \frac{1}{2}\nu^2(R)z^2$$

We find the equation of motion in the z by equating the acceleration to the force per unit mass,

$$\ddot{z} = -\frac{\partial\Phi}{\partial z} = -\nu^2 z.$$

This is the equation for a harmonic oscillator with frequency ν. We conclude that the star oscillates above and below the plane, with some amplitude Z (which could be zero) and period $2\pi/\nu$.