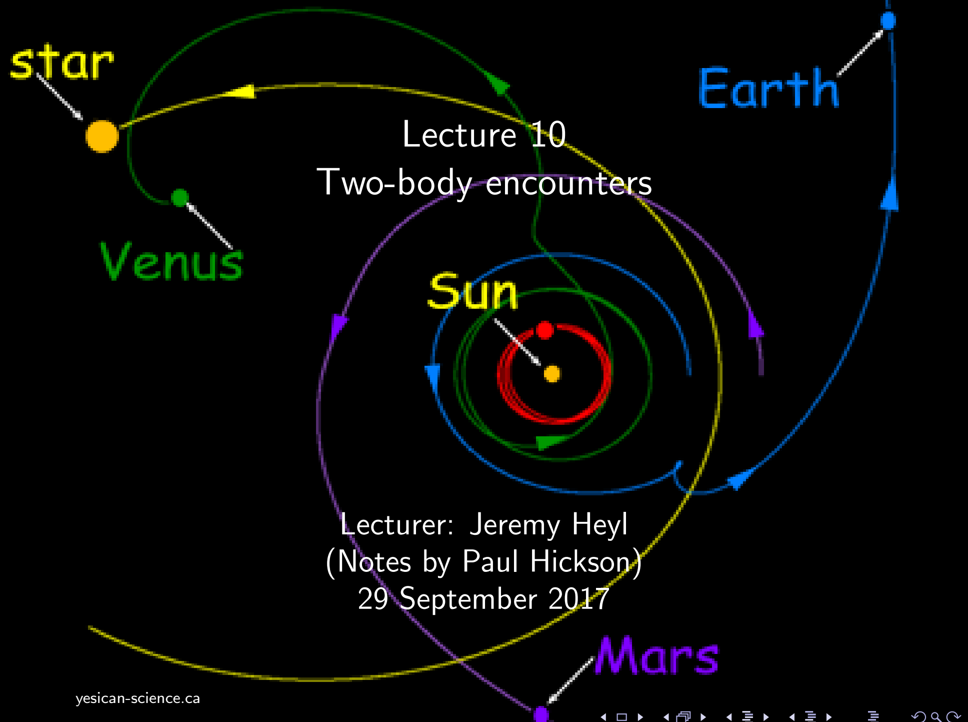

Lecture 10
Two-body encounters

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29 September 2017

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Close encounters

So far we have assumed that stars move on orbits within a smooth gravitational potential, and have ignored the fact that the potential actually has deep minima located at each star. If a star has a close encounter with another, both orbits will be changed.

A *strong encounter* is one that significantly changes the velocity of the star. This will happen if the change in potential energy, at closest approach, is greater than the initial kinetic energy,

$$\frac{Gm^2}{r} \gtrsim \frac{mV^2}{2}$$

which is equivalent to $r \lesssim r_\mathsf{s}$ where

$$r_\mathsf{s} = \frac{2Gm}{V^2}$$

is called the **strong-encounter radius**.

Close encounters

The cross section for strong encounters is then πr_s^2 and the typical time between such encounters is the reciprocal of the collision rate,

$$t_{\mathsf{s}} = \frac{1}{nV\pi r_s^2} = \frac{V^3}{4\pi G^2 m^2 n}.$$

where n is the average number density of stars.

In the vicinity of the Sun, $n \simeq 0.1$ pc^{-3} and stars have random speeds $V \simeq 30$ km/s. For $m = 0.5 M_{\odot}$ we find that $r_{\mathsf{s}} \simeq 1$ AU and $t_{\mathsf{s}} \simeq 10^{15}$ yr.

This time is much longer than the age of the Universe ($\sim 1.4 \times 10^{10}$ yr), so we do not expect close stellar encounters to be a significant factor in the evolution of the galaxy.

Weak encounters

Another possibility is weak encounters that occur when stars pass each other at a greater distance.

The situation is shown in Fig. 3.5. Before the point of closest approach, the star feels a force in the forward (parallel) direction and the perpendicular direction. After this the parallel component of the force reverses, so there is no net change to that component.

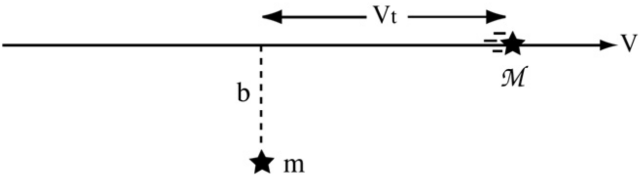

Fig 3.5 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

The perpendicular acceleration of the star is

$$\frac{dV_\perp}{dt} = \frac{Gm}{b^2 + V^2t^2} \cdot \frac{b}{(b^2 + V^2t^2)^{1/2}} = \frac{Gmb}{(b^2 + V^2t^2)^{3/2}}.$$

Weak encounters

Integrating this over time gives the change in the perpendicular component of velocity,

$$\Delta V_\perp = \int_{-\infty}^{\infty} \frac{Gmb}{(b^2 + V^2 t^2)^{3/2}} dt = \frac{2Gm}{bV}.$$

The resulting angular deflection is $\alpha = \Delta V/V = 2GM/bV^2$.

This result applies to a single encounter, but over time the star will have many weak encounters with other stars, with various values of the impact parameter b and will result in deflections in random perpendicular directions.

To evaluate the results of this we need to add the *squares* of the changes in perpendicular velocity, and integrate over all possible impact parameters.

Weak encounters

The rate at which stars are encountered with impact parameter in the range $[b, b+db]$ is $nV(2\pi b)db$, so the rate of change of $\left\langle \Delta V_\perp^2 \right\rangle$ is

$$\frac{d}{dt}\left\langle \Delta V_\perp^2 \right\rangle = \int_{b_{\mathsf{min}}}^{b_{\mathsf{max}}} nV\left(\frac{2Gm}{bV}\right)^2 2\pi b db = \frac{8\pi G^2 m^2 n}{V}\ln\left(\frac{b_{\mathsf{max}}}{b_{\mathsf{min}}}\right).$$

Eventually, the perpendicular component of velocity becomes comparable to the initial velocity and all memory of the initial orbit has been lost. The time required for this to happen is called the **relaxation time** and is given by

$$t_{\mathsf{relax}} = \frac{V^2}{d\left\langle \Delta V_\perp^2 \right\rangle/dt} = \frac{V^3}{8\pi G^2 m^2 n \ln\Lambda} = \frac{t_{\mathsf{s}}}{2\ln\Lambda}.$$

where $\Lambda = b_{\mathsf{max}}/b_{\mathsf{min}}$.

Relaxation time

Typically one takes b_{max} to be the size of the of the entire stellar system and $b_{\text{min}} = r_{\text{s}}$. (The exact values don't matter much since this is inside a logarithm.) For a galaxy, one finds $\ln \Lambda \sim 20$.

We see that the relaxation time for a galaxy is very large, of order 10^{13} yr, which is still much longer than the age of the Universe.

We are therefore justified in ignoring encounters, at least for a large system like a galaxy.

Further insight can be gained by applying the virial theorem. Suppose that the stellar system is spherical with radius R. Then

$$n = 3N/4\pi R^3$$

where N is the total number of stars.

Relaxation time

From the virial theorem,

$$2K + U \simeq NmV^2 - N^2 \frac{Gm^2}{R} = 0$$

so $V^2 \simeq GNm/R$. Putting this into the equation for t_{s} we get

$$t_{\mathsf{s}} = \frac{V^3 R^3}{3G^2 m^2 N} \simeq \frac{NR}{3V} = \frac{N}{3} t_{\mathsf{cr}},$$

where $t_{\mathsf{cr}} = R/V$ is called the **crossing time**, the time taken by a typical star to cross from one side of the galaxy to the other. We see that the strong encounter time is proportional to the number of stars in the system.

Relaxation time

The ratio

$$\Lambda = \frac{R}{r_{\mathsf{s}}} = \frac{V^2 R}{2Gm} \simeq \frac{N}{2},$$

so the relaxation time can now be written as

$$t_{\mathsf{relax}} \simeq \frac{N}{6\ln(N/2)} t_{\mathsf{cr}}.$$

For a typical galaxy with $N \simeq 10^{11}$ stars, relaxation requires $\sim 10^9$ crossing times.

For a globular cluster with 10^6 stars, relaxation will occur in $\sim 10^4 t_{\mathsf{cr}} \simeq 10^{10}$ yr. Relaxation can be important for those systems, particularly in the dense core.

Effects of relaxation

Encounters exchange energy and momentum between stars. Eventually they will approach statistical equilibrium where the velocity distribution is given by the Maxwell-Boltzmann equation,

$$f(v) \propto v^2 \exp\left(-\frac{E}{kT}\right) = v^2 \exp\left\{-\frac{1}{kT}\left[m\Phi(\boldsymbol{x}) + \frac{1}{2}mv^2\right]\right\}.$$

where T is the kinetic "temperature" of the system.

As this happens, some stars will pick up sufficient energy to escape.

The average kinetic energy that a star needs to escape is

$$\left\langle \frac{1}{2}mv_{\mathsf{e}}^2 \right\rangle = -\frac{1}{N}\sum_\alpha m_\alpha \Phi(\boldsymbol{x}_\alpha) = -\frac{2}{N}U = \frac{4}{N}K = 4 \times \frac{3}{2}kT,$$

where the last step follows from the virial theorem. Thus, escape requires only four times the mean kinetic energy.

Evaporation

The fraction of stars that can escape can be found by integrating the velocity distribution

$$\int_{\sqrt{12kT/m}}^{\infty} e^{-mv^2/2kT} v^2 dv \bigg/ \int_0^{\infty} e^{-mv^2/2kT} v^2 dv = \frac{\Gamma(3/2, 6)}{\Gamma(3/2)} \simeq \frac{1}{135.4}.$$

These stars are replenished after a time $t \sim t_{\text{relax}}$, so a substantial fraction of stars will be lost over an **evaporation time** $t_{\text{evap}} \simeq 136\, t_{\text{relax}}$.

For open clusters in the Milky Way, t_{evap} is a few Gyr. In fact these clusters are disrupted sooner than this by tidal forces from spiral arms and molecular clouds.

Encounters also lead to **mass segregation** in star clusters. Massive stars give up energy, to lower-mass stars, and sink toward the centre of the cluster. Similarly, binary stars sink towards the centre.

Mass segregation in the Pleiades

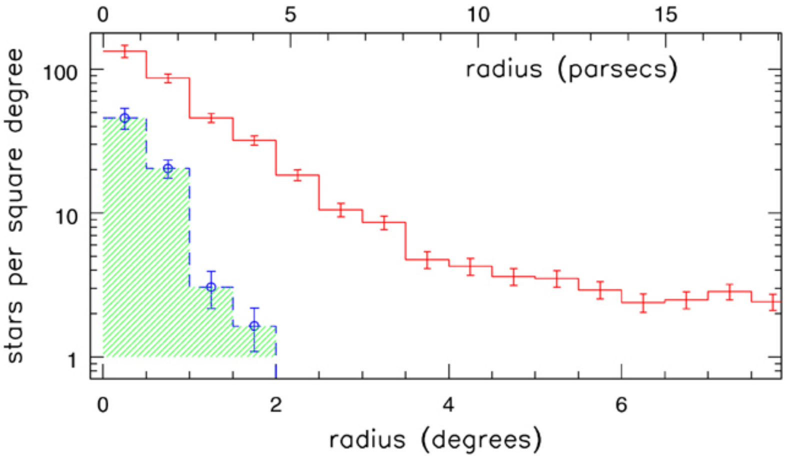

Fig 3.6 (J. Adams) Galaxies in the Universe' Sparke/Gallagher CUP 2007

Radial distribution of stars, green: $M > M_{\odot}$, red: $M < M_{\odot}$.