Lecture 9
Introduction to stellar dynamics

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Newtonian gravity

Except near a massive black hole, the motion of stars in the galaxy is described well by Newton's laws of motion and gravity.

Suppose that we have a number of stars whose masses are m_α, $(\alpha = 1, 2, \cdots)$ located at positions $\boldsymbol{x}_\alpha$. According to Newton, the star at position $\boldsymbol{x}_\alpha$ feels a force per unit mass

$$\boldsymbol{F}_\alpha = -\sum_{\beta \neq \alpha} \frac{G m_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|^3} (\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta).$$

This can be written more simply as

$$\boldsymbol{F}_\alpha(\boldsymbol{x}) = -\boldsymbol{\nabla}\Phi(\boldsymbol{x})$$

where

$$\Phi(\boldsymbol{x}) = -\sum_\alpha \frac{G m_\alpha}{|\boldsymbol{x} - \boldsymbol{x}_\alpha|}, \quad \boldsymbol{x} \neq \boldsymbol{x}_\alpha$$

is the **gravitational potential**.

Potential

Since the number of stars in the galaxy is very large, we can approximate the sum by an integral over the mass density $\rho(\boldsymbol{x})$, averaged over a scale comparable to the separation between stars.

$$\Phi(\boldsymbol{x}) \simeq -\int \frac{G\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x'.$$

The potential satisfies a differential equation. Apply the **Laplacian** operator ∇^2 to both sides,

$$\nabla^2\Phi(\boldsymbol{x}) = -G\int \rho(\boldsymbol{x}')\nabla^2\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\right) d^3x'.$$

The Laplacian of the quantity in parenthesis is equal to zero if $\boldsymbol{x} \neq \boldsymbol{x}'$, so the only contribution to the integral comes when $\boldsymbol{x}' = \boldsymbol{x}$.

Potential

We may therefore replace $\rho(\boldsymbol{x}')$ with $\rho(\boldsymbol{x})$ and take it outside the integral. This gives

$$\nabla^2\Phi(\boldsymbol{x}) = -G\rho(\boldsymbol{x}) \int \nabla^2 \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \right) d^3x'.$$

The integral can be done using Gauss's theorem and has the value -4π. Thus the potential satisfies

$$\nabla^2\Phi(\boldsymbol{x}) = 4\pi G\rho(\boldsymbol{x})$$

which is **Poisson's equation**.

Often, solving Poisson's equation is the simplest way to find the potential associated with a symmetric mass distribution.

Newton's theorems

Newton proved two very useful theorems concerning the force produced by spherically symmetric mass distributions.

- The gravitational force inside a spherical shell of uniform density is zero.
- Outside any spherically symmetric object, the gravitational force is the same as if all its mass had been concentrated at the centre.

Geometric proofs of these theorems were given by Newton and can be found in Sparke and Gallagher. Here we give an alternative proof.

Newton's theorems

By symmetry, the potential can depend only on the distance r from the centre of the shell. Inside and outside the shell, $\rho = 0$, so Poisson's equation becomes

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\Phi}{dr} = 0.$$

Integrating this equation gives

$$\Phi(r) = A + \frac{B}{r},$$

where A and B are constants.

The second term predicts an infinite force at $r = 0$, but we know that the force must vanish there because it is a vector and the problem has spherical symmetry.

Therefore, inside the shell we have $\phi = A$ and $\boldsymbol{F} = -\boldsymbol{\nabla}\Phi = 0$.

Newton's theorems

Outside the shell, we again have

$$\Phi(r) = A + \frac{B}{r},$$

but now the second term can be nonzero.

From the definition of the potential, we see that In the limit as $r \to \infty$, tthe potential must have the limiting form

$$\Phi(r) = -\frac{GM}{r}, \quad r \to \infty.$$

(The shell "looks" like a point mass when seen from far away). Therefore, $B = -GM$ and the force per unit mass

$$\boldsymbol{F} = -\boldsymbol{\nabla}\Phi = -\frac{GM}{r^3}\boldsymbol{r}$$

is the same as for a point mass M located at the centre.

Spherically symmetric mass distribution

From Newton's theorems we see that the force at distance r depends only on the mass $M(r)$ interior to r,

$$\boldsymbol{F} = -\frac{GM(r)}{r^3}\boldsymbol{r}.$$

Therefore, the potential must have the form

$$\Phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r')r'dr'.$$

A star moving in a circular orbit with speed V has acceleration V^2/r, directed inwards. Therefore, the circular velocity is given by

$$V^2 = \frac{GM(r)}{r}.$$

Energy

As a star moves with velocity $\boldsymbol{v}$ through a static potential Φ, the potential at the star's location changes according to

$$\frac{d\Phi}{dt} = \boldsymbol{v} \cdot \boldsymbol{\nabla}\Phi(\boldsymbol{x})$$

The acceleration of a the star is given by Newton's law of motion,

$$\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}\Phi$$

Substituting this in the previous equation we find

$$\frac{d\Phi}{dt} + \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} = 0$$

Therefore,

$$\frac{d}{dt}\left(\Phi + \frac{1}{2}v^2\right) = 0$$

Energy

The quantity in parenthesis is the total energy, per unit mass, of the star

$$E = \frac{1}{2}v^2 + \Phi(\boldsymbol{x})$$

We see that it is conserved (unchanged) as the star moves in its orbit.

This is true as long as the potential at any given point does not change with time. Generally this is a good assumption, unless the galaxy is collapsing, or colliding with another galaxy.

Far from the galaxy, the potential drops to zero. It follows that a star can only escape from the galaxy if its total energy is positive.

To do that it needs to move faster than the local escape speed v_{e}, where

$$v_{\mathsf{e}}^2 = -2\Phi(\boldsymbol{x}).$$

Angular momentum

The angular momentum, per unit mass, of the star is $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{v}$. Its time derivative is

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{x} \times \boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{x} \times \frac{d\boldsymbol{v}}{dt} = -\boldsymbol{x} \times \boldsymbol{\nabla}\Phi.$$

If the system is spherically symmetric, this is zero, so the angular momentum of the star is conserved.

If the system has axial symmetry, only the component L_z along the symmetry axis is conserved.

The *total* angular moment of all the stars in an isolated system is always conserved.

Potential energy

The total potential energy of the system can be found by adding the potential energy of all pairs of stars,

$$U = -\frac{1}{2}\sum_{\alpha,\beta\neq\alpha}\frac{Gm_\alpha m_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|} = \frac{1}{2}\sum_\alpha m_\alpha \Phi(\boldsymbol{x}_\alpha).$$

The factor of $1/2$ is needed because the sum counts each pair twice.

For a continuous distribution, this becomes

$$U = \frac{1}{2}\int \rho\Phi dV.$$

The virial theorem

Consider a cluster of stars with masses m_α and positions $\boldsymbol{x}_\alpha$. Write down the equation of motion and take the dot product with $\boldsymbol{x}_\alpha$,

$$\sum_\alpha \frac{d}{dt}(m_\alpha \boldsymbol{v}_\alpha) \cdot \boldsymbol{x}_\alpha = -\sum_{\alpha,\beta\neq\alpha} \frac{G m_\alpha m_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|^3}(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta)\cdot \boldsymbol{x}_\alpha.$$

Now exchange α and β,

$$\sum_\beta \frac{d}{dt}(m_\beta \boldsymbol{v}_\beta) \cdot \boldsymbol{x}_\beta = -\sum_{\beta,\alpha\neq\beta} \frac{G m_\beta m_\alpha}{|\boldsymbol{x}_\beta - \boldsymbol{x}_\alpha|^3}(\boldsymbol{x}_\beta - \boldsymbol{x}_\alpha)\cdot \boldsymbol{x}_\beta.$$

The left hand sides are the same. If we add the two equations and divide by two we get

$$\begin{aligned}\sum_\alpha \frac{d}{dt}(m_\alpha \boldsymbol{v}_\alpha) \cdot \boldsymbol{x}_\alpha &= -\frac{1}{2}\sum_{\alpha,\beta\neq\alpha} \frac{G m_\alpha m_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|^3}(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta)\cdot(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta),\\ &= -\frac{1}{2}\sum_{\alpha,\beta\neq\alpha} \frac{G m_\alpha m_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|} = U.\end{aligned}$$

The virial theorem

The RHS is the potential energy U. The LHS can be written as

$$\sum_\alpha \frac{d}{dt}(m_\alpha \boldsymbol{v}_\alpha) \cdot \boldsymbol{x}_\alpha = \frac{1}{2}\frac{d^2}{dt^2}\sum_\alpha m_\alpha \boldsymbol{x}_\alpha \cdot \boldsymbol{x}_\alpha - \sum_\alpha m_\alpha \boldsymbol{v}_\alpha \cdot \boldsymbol{v}_\alpha,$$
$$= \frac{1}{2}\frac{d^2 I}{dt^2} - 2K,$$

where $I = \sum_\alpha m_\alpha \boldsymbol{x}_\alpha \cdot \boldsymbol{x}_\alpha$ is the **moment of inertia** of the system and K is its total kinetic energy.

If the overall distribution of mass does not change significantly as the stars move in thier orbits, the moment of inertia is nearly constant and its time derivative is close to zero. In that case our equation becomes

$$2K + U = 0.$$

which is called the **virial theorem**. (Sparke and Gallagher prove a more general version which includes external forces.)