Classical Mechanics 9-7-05

- $F = ma$ → many applications, constraints
- Different types of constraints → different formulations
- Two unifying principles:
  1. The principle of least action. (virtual work) Legendre transform
  2. Hamilton's Equations – phase space

- The principle of least action: easiest route to find EOM of systems with constraints (equations of motion)

Particle Mechanics

$\vec{v} = \frac{d\vec{r}}{dt}$, $\vec{p} = m\vec{v}$, $\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$ (velocity, momentum, force)

$\vec{L}$ = angular momentum = $\vec{r} \times \vec{p}$ and $\vec{N}$ = torque/moment = $\vec{r} \times \vec{F}$

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{N} \quad \therefore \frac{d\vec{L}}{dt} = \underbrace{\cancel{\frac{d\vec{r}}{dt} \times \vec{p}}}_{0} + \vec{r} \times \frac{d\vec{p}}{dt} = N$$

Work

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = m \int \left(\frac{d\vec{v}}{dt} \cdot \vec{v} \right) dt = \frac{m}{2} \int \frac{d}{dt}(v^2)\, dt = \frac{m}{2}(v_2^2 - v_1^2) = T_2 - T_1$$

(kinetic energy) $\boxed{T = \frac{1}{2} m v^2}$

$$\oint \vec{F} \cdot d\vec{s} = 0 \Rightarrow \vec{F} = -\vec{\nabla} V(r)$$

$$W_{12} = V_1 - V_2 \quad \text{so} \quad T_1 + V_1 = T_2 + V_2$$

$$\sum_j \vec{F}_{ji} + \vec{F}_i^{(E)} = \frac{d\vec{p}_i}{dt}$$

$\vec{F}_i^{(E)}$ = external forces acting on i (e.g. gravity)

$\vec{F}_{ji}$ = forces exerted by j on i

$\vec{F}_{ij} = -\vec{F}_{ji}$ (Weak law of action/reaction)

$$\frac{d\vec{p}_i}{dt} = \frac{d}{dt} m v_i = m \frac{d^2 \vec{r}_i}{dt^2} \quad \therefore \quad \frac{d^2}{dt^2} \sum_i m r_i = \sum_i \vec{F}_i^{(E)} + \sum_{i \neq j} \vec{F}_{ji}$$

$$\vec{R} = \frac{\sum m\vec{r}_i}{\sum m} = \frac{\sum m\vec{r}_i}{M} \qquad M = \text{total mass} \quad (\text{i.e. } \therefore \sum m\vec{r}_i = M\vec{R})$$

$$\therefore M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(E)} + \underbrace{\sum_{i\neq j} F_{ji}'}_{0} = \vec{F}^{(E)}$$

(b/c of action/reaction within body)

$$\vec{L} = \vec{r}\times\vec{p} \Rightarrow \sum\left(\vec{r}\times\frac{d\vec{p}_i}{dt}\right) = \sum\frac{d}{dt}(\vec{r}\times\vec{p}_i) = \sum\frac{d\vec{L}_i}{dt}$$

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i\times\vec{F}_i^{(E)} + \sum_{i\neq j}\vec{r}_i\times\vec{F}_{ji}$$

(count each pair once)

$$\sum_{i\neq j}\vec{r}_i\times F_{ji} = \sum_{i<j}(\vec{r}_i\times\vec{F}_{ji} + \vec{r}_j\times\vec{F}_{ij})$$

$$= \sum_{i<j}(\vec{r}_i\times F_{ji} - \vec{r}_j\times F_{ji}) \qquad \text{b/c } \vec{F}_{ij} = -\vec{F}_{ji}$$

(using weak law of action/reaction)

$$= \sum_{i<j}(\vec{r}_i - \vec{r}_j)\times\vec{F}_{ji}$$

9-9-05 STRONG LAW OF ACTION/REACTION

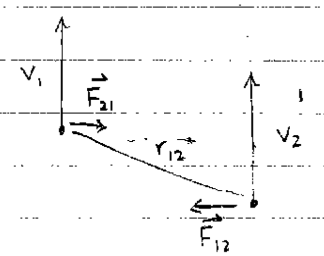
$\vec{F}_{ji} = -\vec{F}_{ij}$ and $\vec{F}_{ij} \parallel (\vec{r}_i - \vec{r}_j)$ [central force]

(the forces between i and j lie along line/rope between i, j

$$\therefore \sum_{i\neq j}\vec{r}_i\times F_{ji} = \sum_{i<j}(\vec{r}_i - \vec{r}_j)\times\vec{F}_{ji} = 0 \qquad \text{b/c } \vec{F}_{ij}\parallel(\vec{r}_i - \vec{r}_j)$$

$$\therefore \dot{L} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i\times\vec{F}_i^{(E)}$$

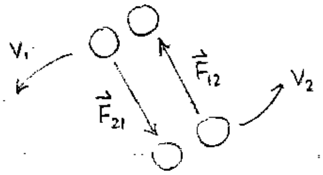
- GRAVITY AND ELECTRIC FORCES ARE CENTRAL
- MAGNETIC FORCES ARE NOT CENTRAL.



∴ NOT CENTRAL

Retarded Gravity

9-9-05



- Force points toward retarded position
  - weak law holds

- Look at the total angular momentum:

Let $\vec{r}_i = \vec{r}_i{}' + \vec{R}$ $\quad$ $\vec{R}$ = position of centre of mass (or displacement)

$\vec{v}_i = \vec{v}_i{}' + \vec{V}$ $\quad$ $\vec{V}$ = velocity of centre of mass

$\vec{r}_i{}'$ = position relative to centre of mass

$\vec{v}_i{}'$ = velocity relative to centre of mass

$$\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i$$
$$= \sum m_i \vec{r}_i{}' \times \vec{v}_i{}' + \sum m_i \vec{r}_i{}' \times \vec{V} + \sum m_i \vec{R} \times \vec{v}_i{}' + \sum m_i \vec{R} \times \vec{V}$$
$$= \sum m_i \vec{r}_i{}' \times \vec{v}_i{}' + (\sum m \vec{r}_i{}') \times \vec{V} + \vec{R} \times \sum m_i \vec{v}_i{}' + \sum m_i \vec{R} \times \vec{V}$$

$\sum m_i \vec{r}_i = (\sum m_i)\vec{R}$ $\quad$ by defn of $\vec{R} = \dfrac{\sum m_i \vec{r}_i}{\sum m_i}$

also $\sum m_i \vec{r}_i = \sum m_i (\vec{R} + \vec{r}_i{}') = \sum m_i \vec{R} + \sum m_i \vec{r}_i{}'$

$\therefore \sum m_i \vec{r}_i{}' = 0$ $\quad$ (sum of all moments about centre of mass $= 0$)

$$0 = \frac{d}{dt} \sum m_i \vec{r}_i{}' = \sum m_i \vec{v}_i{}' = \frac{d}{dt} 0 = 0$$

$$\boxed{\therefore \vec{L} = \underbrace{\sum m_i \vec{r}_i{}' \times \vec{v}_i{}'}_{\text{Internal}} + \underbrace{\sum m_i \vec{R} \times \vec{V}}_{\text{Motion of whole system}}}$$

Energy of an Ensemble

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{s}_i + \sum_{i \neq j} \int_1^2 \vec{F}_{ij} \cdot d\vec{s}_i$$

$$W_{12}^i = \int \vec{F}_i \cdot d\vec{s}_i = \int m \dot{\vec{v}}_i \cdot d\vec{s}_i = \int m \dot{\vec{v}}_i \cdot \vec{v}_i \, dt = \frac{1}{2} m v_i^2 \Big|_0^1$$

$$W_{12} = \sum_i W_{12}^i = \sum \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum \int_1^2 d\left(\frac{1}{2} m v_i^2\right)$$

$$W_{12} = T_2 - T_1 \quad \text{where} \quad T = \frac{1}{2} \sum m v_i^2$$

- In the frame of the centre of mass:

$$T = \frac{1}{2}\sum m(\vec{v}_i' + \vec{V})^2$$
$$= \frac{1}{2}\sum\left[m\vec{v}_i'^2 + 2m\vec{v}_i'\cdot\vec{V} + m\vec{V}^2\right]$$
$$= \frac{1}{2}\sum m\vec{v}_i'^2 + 2\Big(\underbrace{\sum\tfrac{1}{2}m\vec{v}_i'}_{0}\Big)\cdot\vec{V} + \frac{1}{2}\sum m\vec{V}^2$$

$\sum m\vec{v}_i' = \frac{d}{dt}\sum m\vec{r}_i' = \frac{d}{dt}(0) = 0$

$$\therefore T = \frac{1}{2}\sum m\vec{v}_i'^2 + \frac{1}{2}MV^2$$

$\sum m = M$

- Potential Energy

$$\sum_i\int_1^2 F_i^{(e)}\cdot d\vec{s} = -\sum\int_1^2 \nabla_i V_i\cdot d\vec{s}$$
$$= -\sum V_i\Big|_1^2$$

$F_i = \nabla_i V_i$ = gradient of potential for i^{th} particle

Let's assume $V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|)$ → fn of $|\vec{r}_i - \vec{r}_j|$

$$F_{ji} = -\frac{\partial V_{ji}}{\partial r_i} = -\nabla_i V_{ji} = \nabla_j V_{ji} = -F_{ij}$$

9-12-05

$\nabla_{r_i} V_{ij}(|r_i - r_j|) = (r_i - r_j)f$ ← number

gradient of V_{ij} points in direction of $(r_i - r_j)$ → line between i, j

$$\sum_{i\neq j}\int \vec{F}_{ij}\cdot d\vec{s} = \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N}}_{\text{except } i=j}\int \vec{F}_{ji}\cdot d\vec{s}_i = -\sum_{j=1}^{N}\sum_{i=1}^{j-1}\int_1^2\left(\nabla_i V_{ij}\, d\vec{s}_i + \nabla_j V_{ij}\, d\vec{s}_j\right)$$
$$= -\sum_{i<j}\int_1^2\left(\vec{\nabla}_i V_{ij}\cdot d\vec{s}_i + \vec{\nabla}_j V_{ij}\cdot d\vec{s}_j\right)$$

Let $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

so $\nabla_{\vec{r}_{ij}} V_{ij} = \nabla_{\vec{r}_i} V_{ij} = -\nabla_{\vec{r}_j} V_{ij}$ (chain rule)

$d\vec{s}_i$ = displacement of particle i

$d\vec{s}_j$ = displacement of particle j

$\therefore d\vec{s}_i - d\vec{s}_j = d\vec{r}_i + d\vec{r}_j = d\vec{r}_{ij}$

$\nabla_{\vec{r}_i} V_{ij}$ = what happens to potential fn when i is varied

$\nabla_{\vec{r}_j} V_{ij}$ = what happens to potential function when j is varied

variation in

9-12-05

$$\sum_{i\neq j}\int \vec{F}_{ij}\cdot d\vec{s} = -\sum_{i<j}\int_1^2 \left(\nabla_{\vec{r}_{ij}} V_{ij}\cdot d\vec{s}_i - \nabla_{\vec{r}_{ij}} V_{ij}\cdot d\vec{s}_j\right)$$

$$= -\sum_{i<j}\int_1^2 \nabla_{\vec{r}_{ij}} V_{ij}\cdot\left(d\vec{s}_i - d\vec{s}_j\right)$$

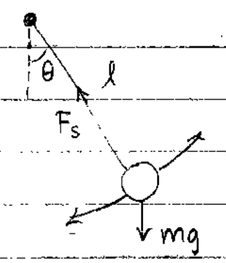
$$= -\sum_{i<j}\int_1^2 \nabla_{r_{ij}} V_{ij}\cdot d\vec{r}_{ij},$$

$$= -\sum_{i<j} V_{ij}\Big|_1^2 = -\frac{1}{2}\sum_{i\neq j} V_{ij}\Big|_1^2$$

Constraints

- There is more to classical mechanics than F=ma.
- Often the motion of a system is constrained in some way.
  - ☆ particles constrained to travel along a curve or surface.

- Pendulum:



$x^2+y^2=\ell^2$

- only need to write down kinetic / potential energies; all forces of constraint vanish

- If we have $f(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, t) = 0$ HOLONOMIC CONSTRAINT

  eg $(\vec{r}_i-\vec{r}_j)^2 - c_{ij}^2 = 0$ for a rigid body c_{ij} = constant

  (for pendulum $\rightarrow \vec{r}^2-\ell^2=0$)

- Could have $r^2-a^2 \geq 0$ NONHOLONOMIC CONSTRAINT

  or some function of positions and velocities vanishes

  eg ball rolling on the floor → many degrees of freedom

- If there are constraints,
  - ↳ the coordinates are no longer independent
  - ↳ forces of constraint not given but must be determined from the solution

↳ if the constraints are holonomic the equations can be used to eliminate some coordinates to get a set of generalized independent coordinates

- These generalized coordinates usually will not come in dyads of triplets that transform as vectors.
  e.g. motion on a sphere $\Rightarrow \theta, \phi$

D'Alembert's Principle

- A virtual displacement is an infinitesimal displacement of the coordinates.
  ↳ $\delta\vec{r}_i$ is consistent with the forces and constraints at a time t

- For each particle we have: $\vec{F}_i = \frac{d\vec{p}_i}{dt}$ $\therefore$ $\vec{F}_i - \frac{d\vec{p}_i}{dt} = 0$

$$\sum\left(\vec{F}_i - \frac{d\vec{p}_i}{dt}\right)\cdot\delta\vec{r}_i = 0$$

- Let's divide $\vec{F}_i$ into applied forces and forces of constraint
  $\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$

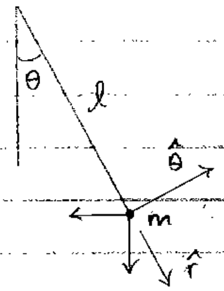
so we have

f_{string} for pendulum

$$\sum\left(\vec{F}_i^{(a)} - \frac{d\vec{p}_i}{dt}\right)\cdot\delta\vec{r}_i + \underbrace{\sum \vec{f}_i\cdot\delta\vec{r}_i}_{\text{Let's look at constraints that do no virtual work}} = 0$$

note: mathematically, $\sum \vec{f}_i \cdot d\vec{r}_i = 0$ since direction of constraint forces $\perp$ direction of motion

Tutorial #1 9-13-05



$$\vec{F}_g = -mg\hat{y}$$
$$\vec{F}_s = f_s\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$
$$\vec{F}_g + \vec{F}_s = \begin{pmatrix} 0 \\ -mg \end{pmatrix} + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} f_s = ma\begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix}$$

- Equate x comp.: $-\sin\theta f = -ma\cos\theta$

$$f = \frac{ma\cos\theta}{\sin\theta}$$

- Equate y comp.: $-mg + \cos\theta f = ma(-\sin\theta)$ $f = \frac{ma\cos\theta}{\sin\theta}$

$$-mg + ma\frac{\cos^2\theta}{\sin\theta} = ma(-\sin\theta)$$

$$\therefore -mg\sin\theta = -ma(\sin^2\theta + \cos^2\theta) = -ma \qquad a = r\ddot{\theta}$$

$$-mr\ddot{\theta} = mg\sin\theta$$

$$\boxed{\ddot{\theta} = -\frac{g}{r}\sin\theta}$$

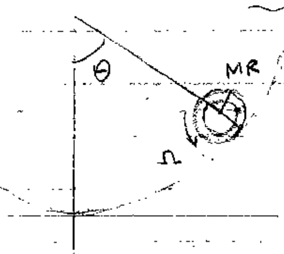
$$\vec{F}_g = mg\cos\theta\hat{r} - mg\sin\theta\hat{\theta} \qquad mr\ddot{\theta}\hat{\theta} = -mg\sin\theta\hat{\theta}$$
$$\vec{F}_s = -f\hat{r} \qquad \vec{F}_c = m\dot{\theta}^2 r\hat{r}$$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 \qquad V = mgh = mg\ell(1-\cos\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \qquad \text{where } L = T - V$$

$$\therefore \frac{d}{dt}(m\ell^2\dot{\theta}) - \frac{\partial}{\partial\theta}(+mg\ell\cos\theta) = m\ell^2\ddot{\theta} + mg\ell\sin\theta = 0$$

$$\therefore m\ell^2\ddot{\theta} = -mg\ell\sin\theta \Rightarrow \boxed{\therefore \ddot{\theta} = -\frac{g\sin\theta}{\ell}}$$



$$T = \frac{1}{2}I\Omega^2 + \frac{1}{2}M(\ell-R)^2\dot{\theta}^2 \qquad \dot{\theta}(\ell-R) + \Omega R = 0$$
$$V = \frac{1}{2}I(\ell-R)(1-\cos\theta) \qquad \Omega = -\frac{\dot{\theta}(\ell-R)}{R}$$
$$T = \frac{1}{2}I\frac{(\ell-R)^2}{R^2}\dot{\theta}^2 + \frac{1}{2}M(\ell-R)^2\dot{\theta}^2$$
$$= \frac{1}{2}(\ell-R)^2\left[M + \frac{I}{R^2}\right]\dot{\theta}^2$$
$$\ddot{\theta}(\ell-R)^2\left[M + \frac{I}{R^2}\right] = -mg(\ell-R)\sin\theta$$

9-14-05

$$\sum \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i \right) \cdot d\vec{r}_i = 0$$ The forces of constraint vanish.

- But the coordinates are still dependent
  ↳ Let's write i - particle label
  j, k - coordinate label, ← some coordinates that are more convenient

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, \ldots, t) \qquad \text{eg. } \vec{r}_i = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y} + v_x t\,\hat{x}$$

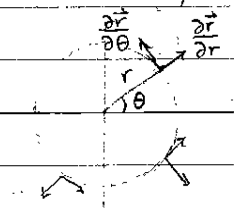
$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_{k=1}^{N} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \underbrace{\frac{\partial \vec{r}_i}{\partial t}}_{\text{change due to coordinate system's change in time}}$$

$$\delta\vec{r}_i = \sum_{k=1}^{N} \frac{\partial \vec{r}_i}{\partial q_k}\delta q_k + \cancel{\frac{\partial \vec{r}_i}{\partial t}}$$ Note: Time is fixed for virtual displacements

Ex. For polar system (r, θ): $\vec{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}$

$$\frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{x} + r\cos\theta\,\hat{y}$$



- $$\sum \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j$$

  i.e. $Q_j = \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$ = "generalized force"

- $$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j$$

- Isolate 1 value of j:

$$\sum_i m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right]$$

b/c $$\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} \right) = m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} + m_i \dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right)$$

9-14-05

$$\therefore \sum_i m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \frac{\partial \vec{r}_i}{\partial q_j} \right) - m \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_j} \right]$$

We know that $\frac{\partial \vec{v}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}$, since $\vec{v}_i = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$

$$\therefore \sum_i m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_j} \right]$$

$$= \sum_i \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \left[\frac{1}{2} m_i v_i^2 \right] \right) - \frac{\partial}{\partial q_j} \left(\frac{1}{2} m_i v_i^2 \right) \right]$$

So we have $\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0$

If the constraints are holonomic then the coordinates can be chosen to be independent ($\sum_j$ is sum of all coordinates q_j)

$$\boxed{\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0}$$ ⇐ often called Lagrange's Equations

$\vec{F} = -\vec{\nabla}_i V$ (conservative forces)

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial V_i}{\partial r_i} \cdot \frac{\partial r_i}{\partial q_j} = -\frac{\partial V_i}{\partial q_j}$$

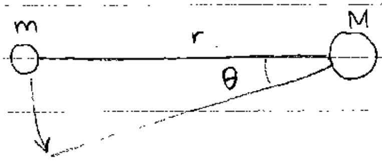
$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j}$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

If $\frac{\partial V}{\partial \dot{q}_j} = 0$ (often the case), $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{where } L = T - V}$

9-14-05

Ex.



$$T = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2$$

$$V = -\frac{GMm}{r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(m r^2 \dot{\theta}) = 0$$

$$\frac{d}{dt}(mr(r\dot{\theta})) = 0$$

$$\frac{d}{dt}(mvr) = 0 \qquad \frac{d}{dt}(L) = 0$$

Noether's Theorem:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 - \frac{GMm}{r^2}\right) = 0$$

The Principle of Least Action 9-19-05

Defn: Action $= S = \int_1^2 L\, dt$, where $L = T - V$

Calculus of Variations

$q_i(t) = q_i^0(t) + \lambda \delta_i(t)$ $\quad$ $q_i^0(t)$ = path particle takes

$\delta_i(t)$ = difference between path and perturbed path

$$\dot{q}_i(t) = \dot{q}_i^0(t) + \lambda \dot{\delta}_i(t)$$

$$S(\lambda) = \int_1^2 L(q_i + \lambda \delta_i, \dot{q}_i + \lambda q_i \dot{\delta}_i, t)\, dt$$

$$\therefore \frac{dS}{d\lambda} = \int_1^2 \left[\frac{\partial L}{\partial q_i} \delta_i + \frac{\partial L}{\partial \dot{q}_i} \dot{\delta}_i\right] dt$$

9-19-05

$$\int_1^2 \frac{\partial L}{\partial \dot{q}_i}\dot{\delta}_i\,dt = \int_1^2 \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \delta_i}{\partial t}\,dt = \int_1^2 \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta_i\right]dt$$

$$= \frac{\partial L}{\partial q_i}\delta_i\Big|_1^2 - \int_1^2 \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta_i\,dt$$

- $\delta_i = 0$ at initial and final time, so 2nd term $= -\int_1^2 \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta_i\,dt$

$$\therefore \frac{dS}{d\lambda} = \int_1^2 \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta_i\,dt = 0 \quad \text{(to minimize } S\text{)}$$

$$\boxed{\therefore \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0}$$

Conserved Quantities:

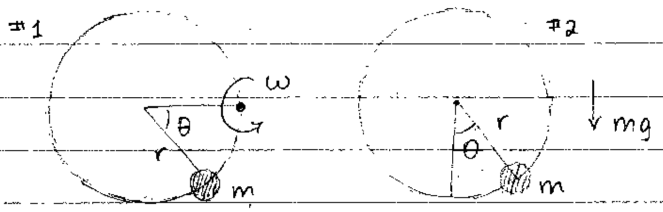
Define: $p_i = \frac{\partial L}{\partial \dot{q}_i}$ = generalized conjugate momentum to q_i

If $\frac{\partial L}{\partial q_i} = 0$, then $\frac{d}{dt}p_i = 0$ (p_i is conserved)

What if $\frac{\partial L}{\partial t} = 0$.

Tutorial #2 9-20-05

3.12 #1 #2



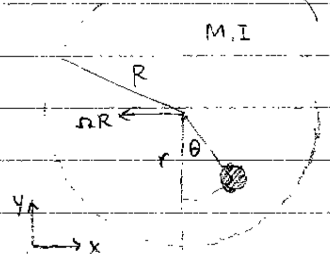
For bead #2 : derive $mg\sin\theta = -m\ddot{\theta}R$

Note that units of g is m/s^2 ⇒ guess that $g \Leftrightarrow \omega^2 R$

∴ guess that eqn. of motion of bead #1 looks like :

$m\omega^2 R\sin\theta = -m\ddot{\theta}R$ (maybe up to a factor)

3.8 Ω $\Omega = \dot{\theta}$ ∴ $V = \Omega R = R\dot{\theta}$



$T_{disk} = \frac{1}{2}M(R\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2$

$\vec{V}_{mass}$ = velocity of wheel + velocity of mass wrt wheel

$= \begin{bmatrix} -R\dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} r\dot{\theta}\cos\theta \\ r\dot{\theta}\sin\theta \end{bmatrix}$ think of it as when $\theta = 0$, $\cos\theta = 1$, $\sin\theta = 0$ all in x-direction

$\therefore V_{mass}^2 = V_x^2 + V_y^2$

$= (-R\dot{\theta} + r\dot{\theta}\cos\theta)^2 + (r\dot{\theta}\sin\theta)^2$

$= R^2\dot{\theta}^2 + r^2\dot{\theta}^2\cos^2\theta - 2R\dot{\theta}^2 r\cos\theta + r^2\dot{\theta}^2\sin^2\theta$

$= R^2\dot{\theta}^2 - 2R\dot{\theta}^2 r\cos\theta + r^2\dot{\theta}^2 = \dot{\theta}^2(r^2 + R^2 - 2Rr\cos\theta)$

$T_{mass} = \frac{1}{2}m\dot{\theta}^2[r^2 + R^2 - 2Rr\cos\theta]$

$\therefore T = \frac{1}{2}\dot{\theta}^2\left[MR^2 + I + mR^2 + mr^2 - 2mRr\cos\theta\right]$ $V = mg(1-\cos\theta)r$

NB. $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \Leftrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = -\frac{\partial V}{\partial\theta} = F_\theta$ = generalized force in θ

↳ eg. $F_q = -\frac{\partial V}{\partial q}$

$\frac{\partial L}{\partial\dot{\theta}} = \dot{\theta}\left[MR^2 + I + mR^2 + mr^2 - 2mRr\cos\theta\right]$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \ddot{\theta}\left[MR^2 + I + mR^2 + mr^2 - 2mRr\cos\theta\right] + \dot{\theta}^2(2Rr\sin\theta)$

9-20-05

$\frac{\partial L}{\partial \theta} = \dot{\theta}^2 mRr\sin\theta - mgr\sin\theta$

$\theta \ll 1$ so $\sin\theta = \theta$, $\cos\theta = 1$, $\theta^2 = 0$

$\therefore \ddot{\theta}[MR^2 + I + mR^2 - 2Rr] = -mgr\theta$

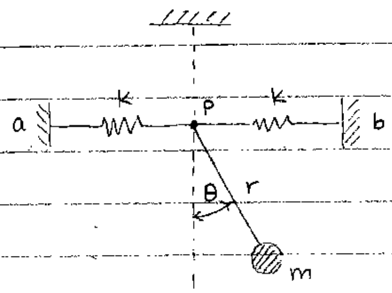
$\theta = A\cos\omega t$

$\ddot{\theta} = -A\omega^2\cos\omega t = -\omega^2\theta$

$\therefore -\omega^2\theta[MR^2 + I + mR^2 - 2Rr] = -mgr\theta$

$$\omega^2 = \frac{mgr}{MR^2 + I + mR^2 - 2Rr}$$

3.7



$\vec{p} = \begin{bmatrix} s \\ 0 \end{bmatrix}$ $\vec{x}_m = \begin{bmatrix} s + r\sin\theta \\ -r\cos\theta \end{bmatrix}$

$\vec{v}_{m_1} = \begin{bmatrix} \dot{s} + r\cos\theta\,\dot{\theta} \\ -r\sin\theta\,\dot{\theta} \end{bmatrix}$

$V = \frac{1}{2}kl_1^2 + \frac{1}{2}kl_2^2 + mg\,y_{mass}$

9-21-05

$p_i = \frac{\partial L}{\partial q_i}$ the generalized momentum

If $\frac{\partial L}{\partial q_i} = 0$, then $\frac{d}{dt} p_i = 0$ p_i is conserved, and $\dot{q}_i$ is a cyclic coordinate

(Defn.) – in other words, if momentum doesn't depend on a particular coordinate q_i (due to symmetry), q_i is a cyclic coordinate and p_i is conserved

↳ for example, if q_i is an angle, and as you rotate the object (and thus change the angle), the system looks the same, an angular momentum is conserved

– What if $\frac{\partial L}{\partial t} = 0$? $L(q_i, \dot{q}_i, t)$

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t}$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$\therefore \frac{dL}{dt} = \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right) + \frac{\partial L}{\partial t}$$

$$= \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\dot{q}_i\right)\right) + \frac{\partial L}{\partial t}$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t} = \frac{d}{dt}\sum_i\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t}$$

$$\therefore \frac{\partial L}{\partial t} = \frac{d}{dt}\left(L - \sum p_i \dot{q}_i\right) \qquad \text{since } p_i = \frac{\partial L}{\partial q_i}$$

(Defn.) – HAMILTONIAN: $-H = L - \sum p_i \dot{q}_i$ is conserved

$L = \frac{1}{2}mv^2 - V(\vec{r})$, $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

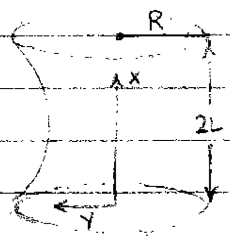
$$\therefore \frac{\partial L}{\partial \dot{q}_i} = \begin{cases} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \frac{\partial L}{\partial \dot{y}} = m\dot{y} \\ \frac{\partial L}{\partial \dot{z}} = m\dot{z} \end{cases}$$

$$\boxed{H = \sum p_i \dot{q}_i - L}$$

$$\sum p_i \dot{q}_i = (m\dot{x})\dot{x} + (m\dot{y})\dot{y} + (m\dot{z})\dot{z} = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = mv^2$$

$$-H = \frac{1}{2}mv^2 - V(\vec{r}) - mv^2 = -\frac{1}{2}mv^2 - V(\vec{r}) = -(T+V) \quad \therefore H = T+V$$

Ex.



Soap bubble between 2 rings

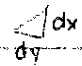
$$A = 2\pi \int y \underbrace{\sqrt{1+(y')^2}\, dx}_{\sqrt{dx^2+dy^2}}$$

each strip:

circum. $= 2\pi y$

"height" $= \sqrt{dx^2+dy^2}$

$= dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$



$d\ell^2 = dx^2 + dy^2$

$L = y(1+(y')^2)^{1/2} \Rightarrow \frac{\partial L}{\partial x} = 0$

$$C = L - \sum \frac{\partial L}{\partial y'} y' = y(1+(y')^2)^{1/2} - \frac{(y')^2 y}{\sqrt{1+(y')^2}}$$

$$\therefore C\sqrt{1+(y')^2} = y(1+(y')^2) - y(y')^2 = y + y(y')^2 - y(y')^2 = y$$

$$\therefore y = C\sqrt{1+(y')^2} \Rightarrow \frac{y}{c} = \sqrt{1+(y')^2} \Rightarrow 1+(y')^2 = \frac{y^2}{c^2}$$

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{c^2} - 1}$$

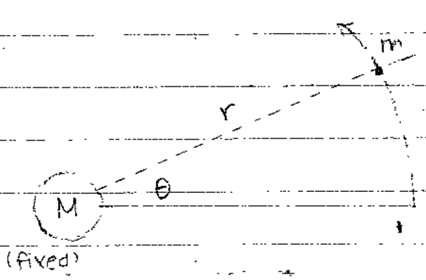
$$y = C\cosh\left(\frac{x-b}{c}\right)$$ hyperbolic cosine

(Aside) $y = \cosh x = \frac{e^x + e^{-x}}{2} \Rightarrow y' = \frac{e^x - e^{-x}}{2}$ $(\cosh ix = \cos x)$

9-23-05

1. Read the problem
2. Draw a picture
3. Identify the degrees of freedom
4. Assign a coordinate to each DOF
5. Calculate T and V : a) sometimes T is difficult, use cartesian or whatever coordinate system
   b) convert to system in 4.
6. Look for cyclic coordinates $\left(\frac{\partial L}{\partial q_i} = 0 \Rightarrow\right.$
   a) Each one gives you a "first integral"
   b) A "First Integral" is an equation for first order, easier than Lagrange (2nd order)
7. If only one coordinate remains, you can either
   a) use Lagrange's equation b) use conservation of H if $\partial L/\partial t = 0$

Ex.



$$T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(r^2\dot\theta^2 + \dot r^2)$$

$$F_r = -\frac{GMm}{r^2}$$

Alternately $\vec{x} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

$$\therefore \vec{v} = \begin{bmatrix} \dot r\cos\theta - r\sin\theta\dot\theta \\ \dot r\sin\theta + r\cos\theta\dot\theta \end{bmatrix} \rightarrow \vec{v}^2 = \dot r^2 + r^2\dot\theta^2$$

$$\therefore T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2)$$

$$V = -\frac{GMm}{r}$$

- θ is a cyclic coordinate $\therefore\ p_\theta = \frac{\partial L}{\partial \dot\theta} = \frac{\partial}{\partial \dot\theta}\left(\frac{1}{2} m r^2 \dot\theta^2\right) = m r^2 \dot\theta$

$$\boxed{\dot\theta = \frac{p_\theta}{m r^2} = \frac{\ell}{r^2}} \qquad \ell = \frac{p_\theta}{m}$$

8. If more than one coordinate remains, it is often easier to use Lagrange's equations for each of them, because the Hamiltonian often couples the DOF.

$$H = \sum_i \frac{\partial L}{\partial \dot q_i}\dot q_i - L = (m r^2 \dot\theta)(\dot\theta) + (m\dot r)(\dot r) - L = m r^2 \dot\theta^2 + m \dot r^2 - L$$

$$H = \frac{1}{2} m (r^2 \dot\theta^2 + \dot r^2) + V = Em \qquad E = \text{constant (in time)}$$

$$\frac{1}{2} r^2 \left(\frac{\ell}{r^2}\right)^2 + \frac{1}{2}\dot r^2 - \frac{GM}{r} = E \Rightarrow \dot r = \sqrt{2E + \frac{2GM}{r} - \frac{\ell^2}{r^2}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot r} = \frac{\partial L}{\partial r}$$

$$\therefore\ m\ddot r = m r \dot\theta^2 - \frac{GMm}{r^2} = m r \left(\frac{\ell}{r^2}\right)^2 - \frac{GMm}{r^2}$$

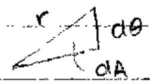
$$\boxed{\therefore\ \ddot r = \frac{\ell^2}{r^3} - \frac{GM}{r^2}}$$

9. You should now have as many equations as DOF. Solve

10. Apply boundary conditions.

9-26-05

$$\ddot{r} = -\frac{GM}{r^2} + \frac{\ell^2}{r^3} \qquad \dot{\theta} = \frac{\ell}{r^2} \qquad \ell = \frac{p_\theta}{m}$$



$$dA = \frac{r^2\,d\theta}{2} \Rightarrow \frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{r^2}{2}\frac{\ell}{r^2} = \frac{\ell}{2}$$

Change of variables: $\frac{df}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt} = \frac{\ell}{r^2}\frac{df}{d\theta} \quad \therefore \frac{d}{dt} = \frac{\ell}{r^2}\cdot\frac{d}{d\theta}$

$\frac{d\left(\frac{1}{r}\right)}{d\theta} = \frac{-1}{r^2}\frac{dr(\theta)}{d\theta} \Rightarrow$ change var. to $u = \frac{1}{r}$ (substitution)

i.e. $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \Rightarrow \frac{dr}{d\theta} = -r^2\frac{du}{d\theta}$. since $u = \frac{1}{r}$

$$\ddot{r} = -\frac{GM}{r^2} + \frac{\ell^2}{r^3} \Rightarrow \frac{d^2r}{dt^2} = \frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{r^2}\frac{dr(\theta)}{d\theta}\right)$$

$$= \frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{r^2}\frac{dr}{d\theta}\right) = -\frac{GM}{r^2} + \frac{\ell^2}{r^3}$$

$$\therefore \frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{r^2}\frac{dr}{d\theta}\right) - \frac{\ell^2}{r^3} = -\frac{GM}{r^2}$$

$$u = \frac{1}{r} \Rightarrow \frac{\ell}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{r^2}\frac{d\left(\frac{1}{u}\right)}{d\theta}\right) - \ell^2u^3 = -GMu^2 \qquad \frac{d\left(\frac{1}{u}\right)}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

$$\frac{\ell}{r^2}\frac{d}{d\theta}\left(-\frac{\ell}{r^2}\frac{1}{u^2}\frac{du}{d\theta}\right) - \ell^2u^3 = -GMu^2 \qquad r^2u^2 = 1$$

$$\ell u^2\frac{d}{d\theta}\left(-\frac{\ell}{1}\frac{du}{d\theta}\right) - \ell^2u^3 = -GMu^2$$

$$-\ell^2u^2\frac{d^2u}{d\theta^2} - \ell^2u^3 = -GMu^2$$

9-26-05

$\therefore \frac{d^2u}{d\theta^2} + u = \frac{GM}{\ell^2}$ $\frac{d^2u}{d\theta^2} + u - \frac{GM}{\ell^2} = 0 \Rightarrow \frac{d^2y}{d\theta^2} + y = 0$

$\Updownarrow$

- Define $y = u - \frac{GM}{\ell^2} \Rightarrow \frac{d^2y}{d\theta^2} + y + \frac{GM}{\ell^2} = \frac{GM}{\ell^2} \Rightarrow \boxed{\frac{d^2y}{d\theta^2} + y = 0}$

$\therefore y = B\cos(\theta - \theta_0)$

$\therefore u = y + \frac{GM}{\ell^2} = \frac{GM}{\ell^2}\left(1 + \underbrace{\frac{B\ell^2}{GM}}_{\varepsilon\,=\,\text{eccentricity}} \cos(\theta-\theta_0)\right) = \frac{1}{r}$

- the orbit is an ellipse: $b = \frac{1}{\sqrt{1-\varepsilon^2}}\,\frac{\ell^2}{GM}$, $a = \frac{1}{1-\varepsilon^2}\,\frac{\ell^2}{GM}$

- the area of the ellipse: $A = \pi ab$

- therefore, period of orbit is: $P = \frac{A}{dA/dt} = \frac{\pi ab}{\ell/2}$

- Kepler's Law: 2nd: equal area traced out in equal time
  (consequence of conservation of angular momentum)

<u>Conservative Systems</u>

- What is V? Is there a V?

Ex. $F_x = \frac{-kx}{(x^2+y^2)^{3/2}}$ $F_y = \frac{-ky}{(x^2+y^2)^{3/2}}$ $(x_0, y_0) \longrightarrow (x, y)$

How much work done?

$$W = \int_{x_0,y_0}^{x,y} (F_x\,dx + F_y\,dy) = \int_{x_0,y_0}^{x,y} \vec{F}\cdot d\vec{s}$$

$$\therefore\ W = \int_{x_0,y_0}^{x,y} \frac{-kx}{(x^2+y^2)^{3/2}}\,dx - \frac{ky}{(x^2+y^2)^{3/2}}\,dy$$

If $\dfrac{\partial f}{\partial y} = \dfrac{-ky}{(x^2+y^2)^{3/2}}$ what is f? $\quad f = \dfrac{k}{(x^2+y^2)^{1/2}}$

If $\dfrac{\partial f}{\partial x} = \dfrac{-kx}{(x^2+y^2)^{3/2}}$ what is f? $\quad f = \dfrac{k}{(x^2+y^2)^{1/2}}$

$$\therefore\ W = \int_{x_0,y_0}^{x,y} \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \right)$$ total derivative

$$= \int_{x_0,y_0}^{x,y} d(f)$$

$$= f(x,y) - f(x_0,y_0)$$

$$\therefore\ W = \frac{k}{(x^2+y^2)^{1/2}} - \frac{k}{(x_0^2+y_0^2)^{1/2}}$$ (no path dependence)

$V = -f = -\dfrac{k}{r}$ conservative force

Ex. Friction (non-conservative) $F = \mu mg$ $\quad F_x = -\mu mg \dfrac{dx}{ds}$

$\therefore\ W = -mg\mu l$

Ex. $F_x = 3Bx^2y^2$ Conservative? Yes

$F_y = 2Bx^3y$

Ex. $F_x = axy$ Conservative? NO

$F_y = bxy$

Tutorial #3 9-27-05

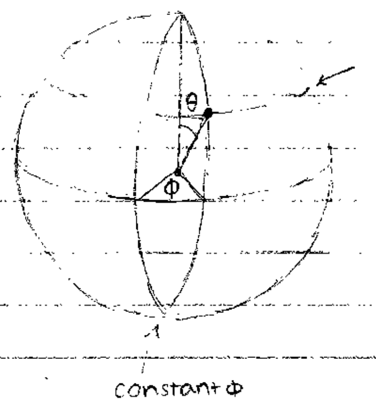
- Iff $\delta S = \int \delta L \, dt = 0$, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

· $\delta L = ?$ $\Rightarrow$ $q(t) = q^{soln}(t) + \lambda\,\delta(t)$ $x^{soln}(t) = A\sin\omega t$

$x(t) = A\sin\omega t + \varepsilon \sin\omega t$

$\delta S = \int \delta L \, dt = 0 \iff \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$

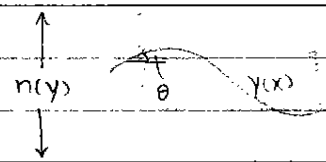
δL defined as difference true path and perturbed path



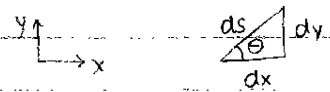
- equator : $\theta = \pi/2$

- find for points on the sphere (equidistant from centre) lie on the same plane

Ex.



- Fermat's Principle: light takes the path requiring the least amount of time



$dt = \frac{ds}{v} = \frac{n}{c} ds = \frac{n}{c}\left(1+(y')^2\right)^{1/2} dx$

$ds = \sqrt{dx^2 + dy^2}$

$= dx\sqrt{1 + dy^2/dx^2}$

$= dx\sqrt{1+(y')^2}$

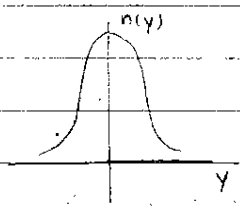
$\therefore t = \frac{1}{c}\int n\sqrt{1+(y')^2}\, dx$

$L = n\sqrt{1+(y')^2} \rightarrow H = \frac{\partial L}{\partial y'} y' - L$ (does $\frac{\partial L}{\partial x} = 0$? yes $\therefore$ x is cyclic)

$= \frac{ny'}{\sqrt{1+(y')^2}} y' - n\sqrt{1+(y')^2}$

$$\therefore H = \frac{n}{\sqrt{1+(y')^2}}\left((y')^2 - (1+(y')^2)\right) = \frac{-n}{\sqrt{1+(y')^2}}$$

$$y' = \tan\theta \quad \therefore H = \frac{-n}{\sqrt{1+\tan^2\theta}} = \frac{-n}{\sqrt{\sec^2\theta}} = -n\cos\theta$$

if 

→ path gets more and more horizontal

a) Lagrange's Equation

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \Rightarrow \frac{d}{dx}\left(\frac{ny'}{\sqrt{1+(y')^2}}\right) - \frac{\partial n}{\partial y}\sqrt{1+(y')^2} = 0$$

$$\frac{d}{dx}\left[\frac{n(y)\,y'}{\sqrt{1+(y')^2}}\right] = \left(\frac{\partial n}{\partial y}y'\right)\left(\frac{y'}{\sqrt{1+(y')^2}}\right) - \frac{ny'y'y''}{\sqrt{1+(y')^2}} - \frac{\partial n}{\partial y}\sqrt{1+(y')^2} = 0$$

$$= \frac{\partial n}{\partial y}\left([1+(y')^2](y')^2 - [1+(y')^2][1+(y')^2]\right)$$

$$+ ny''\left[(1+(y')^2) - (y')^2\right] = 0$$

$$\therefore \frac{\partial n}{\partial y}(1+(y')^2) + ny'' = 0$$

$$\therefore y'' = \frac{\partial \ln n}{\partial y}(1+(y')^2) \qquad y' = \tan\theta$$

$$\theta' = \frac{\partial \ln n}{\partial y}$$

9-28-05

- In general, $V = -\int_{x_i^0 y_i^0 z_i^0}^{x_i, y_i, z_i} \sum_{i=1}^{P} \left(F_{x_i} dx_i + F_{y_i} dy_i + F_{z_i} dz_i \right)$

$F_{x_i} = -\frac{\partial V}{\partial x_i}$ Let's calculate $\frac{\partial F_{x_3}}{\partial y_4}$,

$F_{y_i} = -\frac{\partial V}{\partial y_i}$ $F_{x_3} = -\frac{\partial V}{\partial x_3}$ $\frac{\partial F_{x_3}}{\partial y_4} = \frac{\partial F_{y_4}}{\partial x_3}$

$F_{z_i} = -\frac{\partial V}{\partial z_i}$ $F_{y_4} = -\frac{\partial V}{\partial y_4}$ $-\frac{\partial^2 V}{\partial x_3 \partial y_4} = -\frac{\partial^2 V}{\partial y_4 \partial x_3}$

Ex. $F_x = 3Bx^2y^2$ $\Rightarrow$ $\frac{\partial F_x}{\partial y} = 6Bx^2y$ $\frac{\partial F_y}{\partial x} = 6Bx^2y$ ∴ conservative
$F_y = 2Bx^3y$

$$V = \int^x F_x\, dx = \int^x 3Bx^2y^2\, dx = Bx^3y^2 + C(y)$$
$$= \int^y F_y\, dy = \int^y 2Bx^3y\, dy = Bx^3y^2 + C(x)$$

- If the force is conservative, V can be constructed given F

- Back to Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = F_{q_i} = -\frac{\partial V}{\partial q_i} + F_{q_i}^{NC}$$ → non-conservative

If $\frac{\partial V}{\partial \dot{q}_i} = 0$ then $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_{q_i}^{NC}$

Ex. Central Force

$$\vec F_{2,1} = \vec F_{1,2}$$

$$\vec F_{2,1} \parallel \vec r_1 - \vec r_2$$

m_2 $\vec r_2$

Let $\vec r = \vec r_1 - \vec r_2$, $F(|\vec r_1 - \vec r_2|)$

m_1 $\vec r_1$

$$V = V(|\vec r_1 - \vec r_2|) = V(|\vec r|)$$

- if force points in direction of $\vec r$ and only depends on $|\vec r|$, potential V is conservative

$$L = \frac{1}{2} m_1 \dot{\vec r}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec r}_2^{\,2} - V(|\vec r_1 - \vec r_2|)$$

Let $\vec R = \dfrac{m_1 \vec r_1 + m_2 \vec r_2}{m_1 + m_2}$ and $\vec r = \vec r_1 - \vec r_2$

$$\vec r_1 = \vec R + \frac{m_2 \vec r}{m_1 + m_2} \qquad \vec r_2 = \vec R - \frac{m_1 \vec r}{m_1 + m_2}$$

Check: $m_1 \vec r_1 + m_2 \vec r_2 = (m_1 + m_2)\vec R + \dfrac{m_1 m_2 \vec r}{m_1 + m_2} - \dfrac{m_1 m_2 \vec r}{m_1 + m_2}$ ✓

Let $\vec V = \dfrac{d\vec R}{dt} = \dot{\vec R}$

$\vec v = \dfrac{d\vec r}{dt} = \dot{\vec r}$

$$L = \frac{1}{2} M \vec V^2 + \frac{1}{2} m_1 \left(\frac{m_2 \vec v}{m_1 + m_2} \right)^2 + \frac{1}{2} m_2 \left(\frac{m_1 \vec v}{m_1 + m_2} \right)^2 - V$$

$$= \frac{1}{2} M \vec V^2 + \frac{1}{2} \vec v^2 \left(\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2} \right) - V$$

$$L = \frac{1}{2} M \vec V^2 + \frac{1}{2} \left(\frac{m_1 m_2}{m_1 + m_2} \right) \vec v^2 - V(r)$$

$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ = reduced mass

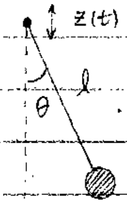
$$= \frac{1}{2} M \vec V^2 + \frac{1}{2} \mu \vec v^2 - V(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec V} \right) = 0 \Rightarrow \frac{\partial L}{\partial \vec V} = M \vec V = \text{constant}$$

Concept of Mechanical Similarity

↳ if you know that underlying eqns are same, you already know the solution

Ex. Jiggly Pendulum



$$\vec{x}_{bob} = \begin{bmatrix} \ell \sin\theta \\ z - \ell\cos\theta \end{bmatrix} \Rightarrow \vec{v}_{bob} = \begin{bmatrix} \ell\cos\theta\,\dot\theta \\ \dot z + \ell\sin\theta\,\dot\theta \end{bmatrix}$$

$$T = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m (\ell^2 \cos^2\theta\,\dot\theta^2 + \dot z^2 + \ell^2 \sin^2\theta\,\dot\theta^2 + 2\dot z \ell \sin\theta\,\dot\theta)$$

$$= \frac{1}{2} m (\ell^2 \dot\theta^2 + \dot z^2 + 2\dot z \ell \sin\theta\,\dot\theta)$$

$V =$

Ex. Charge in E/M Field $\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$ $\vec\nabla\cdot\vec{B} = 0$ $\vec\nabla\cdot(\vec\nabla\times\vec{A}) = 0$

$\therefore \vec{B} = \vec\nabla\times\vec{A}$

↳ Although in certain cases $\vec{E} = -\vec\nabla\phi$, this isn't true in general $\left(\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}\right)$

↳ Force depends on velocity, but we can still write a Lagrangian.

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$\vec\nabla\cdot\vec{B} = 0$ and $\vec\nabla\cdot(\vec\nabla\times\vec{A}) = 0$ $\therefore \vec{B} = \vec\nabla\times\vec{A}$

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times (\vec\nabla\times\vec{A})\right)$$

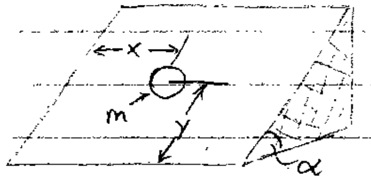
Vector Triple Product:

$A \times (B \times C) = B(A\cdot C) - C(A\cdot B)$

$$\therefore \vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times (\vec{\nabla}\times\vec{A})\right) = q\left(\vec{E} + \frac{1}{c}[\nabla(\vec{v}\cdot\vec{A}) - \vec{A}(\nabla\vec{v})]\right)$$

9-30-05

Ex. Frictional Forces $f = \mu m g \cos\alpha$ (total friction)



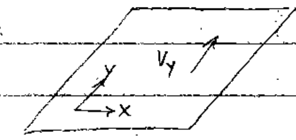
(1) $f_x = \dfrac{-f\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$ $f_y = \dfrac{-f\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$

- portion of velocity in x/y directions (since friction goes against velocity)

so $m\ddot{x} = -\mu m g \cos\alpha \left(\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)$

$m\ddot{y} = -\mu m g \cos\alpha \left(\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) + \underbrace{m g \sin\alpha}_{\text{force of gravity}}$

- A related example:



- conveyor belt moving at V_y in +ve direction

$f = \mu m g$ (1) $f_x = \dfrac{-f\dot{x}}{\sqrt{\dot{x}^2+(V_y-\dot{y})^2}}$ (2) $f_y = \dfrac{-f\dot{y}}{\sqrt{\dot{x}^2+(V_y-\dot{y})^2}}$

c.f. force across belt with belt moving or not moving (take $\dot{y}=0$)

$f_x^M = \dfrac{-f\dot{x}}{\sqrt{\dot{x}^2+(V_y)^2}}$ v. $f_x^S = -f$

$\boxed{f_x^M \ll f_x^S}$

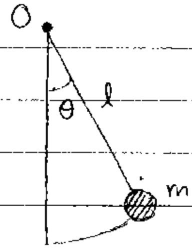
eg. open champagne: twisting motion reduces the friction against pulling out motion

Viscous Forces 10-03-05

$$\delta W_{total} = -\sum a_i(\dot{x}_i\,\delta x_i + \dot{y}_i\,\delta y_i + \dot{z}_i\,\delta z_i)$$

$$= \sum F_{qr}\,\delta q_r \qquad W_{Total} = \sum \int F_{q_r}\,dq_r$$

↑ generalized force



$$x = l\sin\theta \qquad \dot{x} = l\cos\theta\,\dot{\theta}$$

$$y = -l\cos\theta \qquad \dot{y} = l\sin\theta\,\dot{\theta}$$

$$\left.\begin{aligned} F_x &= -a\dot{x} = -al\cos\theta\,\dot{\theta} \\ F_y &= -a\dot{y} = -al\sin\theta\,\dot{\theta}\end{aligned}\right\} \quad F_\theta = F_x\left(\frac{\partial x}{\partial\theta}\right) + F_y\left(\frac{\partial y}{\partial\theta}\right)$$

$$\therefore F_\theta = (-al\cos\theta\,\dot{\theta})(l\cos\theta) + (-al\sin\theta\,\dot{\theta})(l\sin\theta)$$

$$= -al^2\cos^2\theta\,\dot{\theta} - al^2\sin^2\theta\,\dot{\theta}$$

$$= -al^2\dot{\theta}\,(\cos^2\theta + \sin^2\theta)$$

$$\boxed{F_\theta = -al^2\dot{\theta}} \quad \Rightarrow \text{ has units of torque}$$

"generalized force corresponds to θ direction — a torque"

$$L = T - V = \frac{1}{2}mv^2 - mgy = \frac{1}{2}ml^2\dot{\theta}^2 - l\cos\theta\,(mg)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = -al^2\dot{\theta}$$

$$\frac{d}{dt}(ml^2\dot{\theta}) - (-l\sin\theta)mg = -al^2\dot{\theta}$$

$$ml^2\ddot{\theta} - l\sin\theta\,(mg) = -al^2\dot{\theta} \quad \Rightarrow \quad ml^2\ddot{\theta} + \theta mgl + al^2\dot{\theta} = 0$$

- If $\dfrac{\partial F_x}{\partial y} = \dfrac{\partial F_y}{\partial x}$, $F_x = -\dfrac{\partial\Phi}{\partial x}$

- $F_{q_r} = \dfrac{\partial P}{\partial\dot{q}_r}$ We can often calculate the force in terms of the power function (dissipative forces)

↳ P (the power function) has units of power

(N · m/s = J/s = W)

10-03-05

- Analogous to $F_{q_r} = -\frac{\partial V}{\partial q_r}$ (conservative forces)

Defn: - The Power Function P:

$$P = \int \sum_{i=1}^{p} \left(f_{x_i} d\dot{x}_i + f_{y_i} d\dot{y}_i + f_{z_i} d\dot{z}_i \right)$$

- If $\frac{\partial f_{x_i}}{\partial \dot{y}_k} = \frac{\partial f_{y_k}}{\partial \dot{x}_i}$ for all combinations, then P is independent of path

$$\therefore f_{x_i} = \frac{\partial P}{\partial \dot{x}_i}, \text{ etc.}$$

- $$F_{q_r} = \sum_{i=1}^{p} \left(f_{x_i} \frac{\partial x_i}{\partial q_r} + f_{y_i} \frac{\partial y_i}{\partial q_r} + f_{z_i} \frac{\partial z_i}{\partial q_r} \right)$$

$$= \sum_{i=1}^{p} \left(f_{x_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_r} + f_{y_i} \frac{\partial \dot{y}_i}{\partial \dot{q}_r} + f_{z_i} \frac{\partial \dot{z}_i}{\partial \dot{q}_r} \right)$$

$$= \sum_{i=1}^{p} \left(\frac{\partial P}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \dot{q}_r} + \frac{\partial P}{\partial \dot{y}_i} \frac{\partial \dot{y}_i}{\partial \dot{q}_r} + \frac{\partial P}{\partial \dot{z}_i} \frac{\partial \dot{z}_i}{\partial \dot{q}_r} \right)$$

$$F_{q_r} = \frac{\partial P}{\partial \dot{q}_r}$$

$$f_i = \phi_i(x_i, y_i, z_i, v_i, t)$$

$$f_{x_i} = \frac{\dot{x}_i \phi}{v_i} \qquad f_{y_i} = \frac{\dot{y}_i \phi}{v_i}$$

$$\frac{\partial f_{x_i}}{\partial \dot{y}_i} = \dot{x}_i \frac{\partial}{\partial v_i}\left(\frac{\phi}{v_i}\right)\frac{\dot{y}_i}{v_i} \qquad \frac{\partial f_{y_i}}{\partial \dot{x}_i} = \dot{y}_i \frac{\partial}{\partial v_i}\left(\frac{\phi}{v_i}\right)\frac{\dot{x}_i}{v_i}$$

Tutorial #4 10-4-05

5.5 $V = \frac{1}{2}k(\ell - \ell_0)^2 \qquad \ell^2 = x^2 + s^2$

$$\therefore V = \frac{1}{2}k\left(\sqrt{x^2+s^2} - \ell_0\right)^2 = \frac{1}{2}k\left(s\sqrt{1+\frac{x^2}{s^2}} - \ell_0\right)^2$$

$(1+x)^n \approx 1 + nx$

$$\approx \frac{1}{2}k\left(s\left(1+\frac{x^2}{2s^2}\right) - \ell_0\right)^2$$

$\log(1+x) \approx x$

$$= \frac{1}{2}k\left(s + \frac{x^2}{2s} - \ell_0\right)^2$$

$$= \frac{1}{2}k(s-\ell_0)^2 + \frac{k(s-\ell_0)x^2}{2s} + \frac{kx^4}{8s}\left(\ell_0\right)$$ ← missing

Taylor expansion:

$$V = \frac{1}{2}k\left(s\sqrt{1+\frac{x^2}{s^2}} - \ell_0\right)^2 \qquad \text{let } u = \frac{x^2}{s^2} \text{ (small quantity)}$$

$$= \frac{1}{2}k(s-\ell_0) + u\frac{\partial V}{\partial u}\bigg|_{u=0} + \frac{1}{2}u^2\frac{\partial^2 V}{\partial u^2} + \ldots$$

$$\frac{\partial V}{\partial u} = \frac{1}{2}k\,2\left(s\sqrt{1+u^2} - \ell_0\right)\left(\frac{s}{2\sqrt{1+u}}\right) = \frac{k(s-\ell_0)s}{2}$$

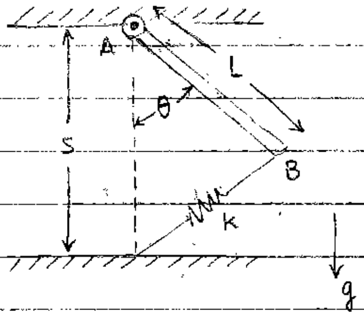
$$= \frac{k}{2}\left(s^2 - \frac{\ell_0 s}{\sqrt{1+u}}\right)$$

$$\frac{\partial^2 V}{\partial u^2} = \frac{k}{2}\left(+\frac{1}{2}\right)\frac{\ell_0 s}{(1+u)^{3/2}} = \frac{k\ell_0 s}{4}$$

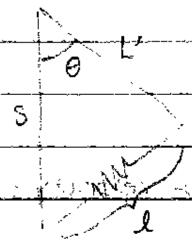
$$V = \frac{ks(s-\ell_0)}{2}\frac{x^2}{s^2} + \frac{k\ell_0 s}{4}\left(\frac{1}{2}\left(\frac{x^2}{s^2}\right)^2\right)$$

$$= \frac{k(s-\ell_0)}{2s}x^2 + \frac{k\ell_0}{8s^3}x^4$$ b/c 1st const term DOESN'T MATTER

5.6



$$V = V_{spring} + V_{gravity}$$
$$= \frac{1}{2}k(l-l_0)^2 - mg\cos\theta\left(\frac{L}{2}\right)$$



Cosine Law:
$$l^2 = s^2 + L^2 - 2sL\cos\theta$$

$$\therefore\ V = \frac{1}{2}k\left(\sqrt{s^2+L^2-2sL\cos\theta} - l_0\right)^2 - \frac{1}{2}mgL\cos\theta$$

$\theta \ll 1$ (small) $\therefore\ \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\theta^2} = 1 - \frac{\theta^2}{2}$

$$\therefore\ V = \frac{1}{2}k\left(\sqrt{s^2+L^2-2sL\left(1-\frac{\theta^2}{2}\right)} - l_0\right)^2 - \frac{1}{2}mgL\left(-\frac{\theta^2}{2}\right)$$
$$= \frac{1}{2}k\left(\sqrt{s^2+L^2-2sL+sL\theta^2} - l_0\right)^2 + \frac{1}{2}mgL\frac{\theta^2}{2}$$
$$= \frac{1}{2}k\left[(s-L)\sqrt{1+\frac{sL\theta^2}{s^2+L^2}} - l_0\right]^2 + \frac{1}{2}mgL\frac{\theta^2}{2}$$
$$= \frac{1}{2}k\Big[(s-L)$$

The Power Function

10-5-05

$$P = \int \sum_{i=1}^{\infty} \frac{\phi_i}{v_i} \left(\dot{x}_i d\dot{x}_i + \dot{y}_i d\dot{y}_i + \dot{z}_i d\dot{z}_i \right) \qquad (*)$$

$f_i = \phi_i(x_i, y_i, z_i, v_i, t)$ $\qquad \vec{F}_i = \frac{\vec{v}_i}{|\vec{v}_i|} f_i$ (Force along direction of the velocity)

$(v_i = |\vec{v}_i|)$ $\qquad \vec{F}_i \parallel \vec{v}_i$

Note: $\frac{\dot{x}_i}{v_i}$ = portion of velocity that is in the x-direction

$$P = \int \sum_{i=1}^{p} \phi_i \, dv_i$$

- Assume $f_i = a_i v_i^n \propto$ "some power" of velocity

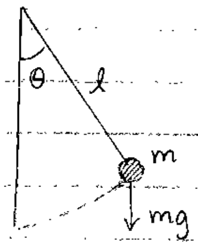
$$\frac{f_i}{a_i v_i^n} \Rightarrow \frac{1}{n+1} \frac{P}{a_i v_i^{n+1}}$$

$$a_i \Rightarrow a_i v_i$$

$$a_i v_i \Rightarrow \frac{1}{2} a_i v_i^2 \quad \text{(Rayleigh Dissipation Function)}$$

↳ how quickly energy is dissipated as a particle moves through a fluid

Ex. Redo the pendulum problem.



→ as the pendulum moves, there is some viscous force against its motion ($\propto v$)

$\therefore f = av$ (where $a < 0$)

$\therefore P = \frac{1}{2} a v^2$ $v = \ell\dot{\theta}$

$T = \frac{1}{2} m\ell^2\dot{\theta}^2$ $V = -mg\ell\cos\theta$ Can't include viscous force in V b/c it is not conservative

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{\partial P}{\partial \dot{\theta}}$ $L = \frac{1}{2} m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$

$\frac{d}{dt}(m\ell^2\dot{\theta}) - (-mg\ell\sin\theta) = \frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}a\ell^2\dot{\theta}^2\right) = a\ell^2\dot{\theta}$

$\therefore m\ell^2\ddot{\theta} + mg\ell\sin\theta = a\ell^2\dot{\theta}$

Ex. $F_x = -ay$ $F_y = ax$

a) Is it conservative? $\frac{\partial F_x}{\partial y} = -a$ $\frac{\partial F_y}{\partial x} = a$ $\frac{\partial F_x}{\partial y} \neq \frac{\partial F_y}{\partial x}$

or $\nabla \times \vec{F} = 2a\hat{z}$

$\therefore$ Not conservative $\Rightarrow$ cannot build a potential!

$\frac{\partial F_x}{\partial \dot{y}} = 0$ $\frac{\partial F_y}{\partial \dot{x}} = 0$ $\therefore$ can build a power function (F_x, F_y don't depend on velocity) $\frac{\partial F_x}{\partial \dot{y}} = \frac{\partial F_y}{\partial \dot{x}}$

$\therefore P = ax\dot{y} - ay\dot{x}$

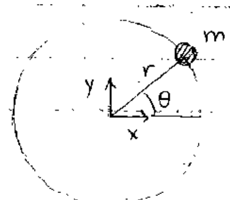
$= a(r\cos\theta)(r\cos\theta\,\dot{\theta})$

$- a(r\sin\theta)(-r\sin\theta\,\dot{\theta})$

$= ar^2\cos^2\theta\,\dot{\theta} + ar^2\sin^2\theta\,\dot{\theta}$

$P = ar^2\dot{\theta}$

$F_\theta = ar^2$



$x = r\cos\theta$ $\dot{x} = -r\sin\theta\,\dot{\theta}$

$y = r\sin\theta$ $\dot{y} = r\cos\theta\,\dot{\theta}$

$T = \frac{1}{2} m r^2 \dot{\theta}^2 \qquad V = 0$

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\partial P}{\partial \dot{\theta}} \qquad P = a r^2 \dot{\theta}$

$\therefore m r^2 \ddot{\theta} - 0 = a r^2$

$$\boxed{\ddot{\theta} = \frac{a}{m}}$$

What if there is viscosity?

$P = -\frac{1}{2} b v^2$

$\therefore P_{\text{whole system}} = a r^2 \dot{\theta} - \frac{1}{2} b r^2 \dot{\theta}^2 \Rightarrow \frac{\partial P}{\partial \dot{\theta}} = a r^2 - b r^2 \dot{\theta}$

$$\boxed{\therefore m r^2 \ddot{\theta} = a r^2 - b r^2 \dot{\theta}}$$

$\ddot{\theta} = 0$ (no acceleration) when $\boxed{\dot{\theta} = \frac{a}{b}}$ $\boxed{v = \frac{a r}{b}}$

$\therefore$ eventually stops accelerating at const. velocity of $\frac{a r}{b}$

$q\vec{v} = q\left(\frac{a r}{b}\right)$ - physically, this is a metal loop with an increased magnetic field through the middle which induces a current through loop

$\dot{J} = q \frac{1}{2} (\vec{\nabla} \times \vec{E}) r$

$\dot{\theta} = \frac{a}{b}\left(1 - e^{-\frac{b}{m}t}\right)$ - resistivity is like viscosity to the current

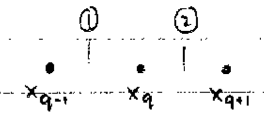
slight delay until reaching "terminal velocity"

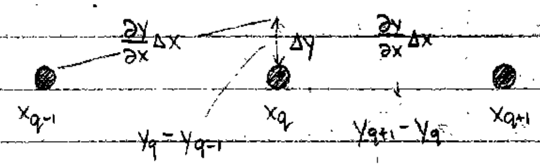
10-7-05

$$V = \frac{1}{2}\sum_{r=1}^{n} \frac{\tau}{a}(y_{r+1}-y_r)^2 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_q} = m\ddot{y}_q$$

$$T = \frac{1}{2}\sum_{r=1}^{n} m\dot{y}_r^{\,2} \qquad \frac{\partial L}{\partial y_q} = -\frac{\tau}{a}\underbrace{(y_{q+1}-y_q)}_{\frac{\partial y}{\partial x}\Delta x \text{ ②}}(-1) - \frac{\tau}{a}\underbrace{(y_q - y_{q-1})}_{①}$$

$$= \frac{\tau}{a}(y_{q+1} - 2y_q + y_{q-1})$$

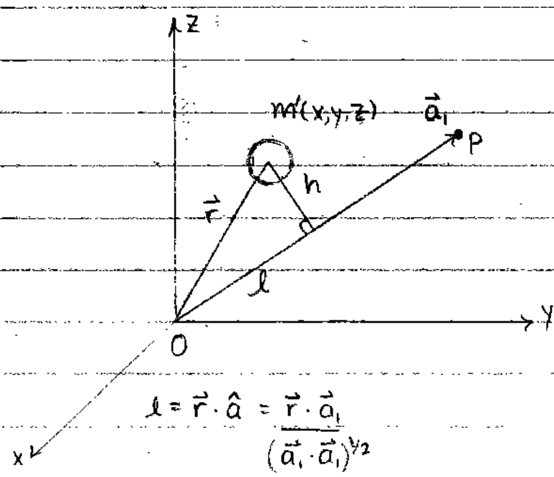




$$m\ddot{y} = \frac{\tau}{a}a^2\frac{\partial^2 y}{\partial x^2}$$ wave equation

$$y = \sin\left(\frac{\pi x}{L}\right)\sin(\omega t) \qquad f = \frac{1}{2}\sqrt{\frac{\tau}{\mu}}\frac{1}{L} \qquad \boxed{\mu = \frac{M}{L}}$$

Moments of Inertia



What is the moment of inertia about $\overline{OP}$?

$$I_{OP} = \sum m'h^2$$
$$= \sum m'(|\vec{r}|^2 - \ell^2)$$
$$= \sum m'\left(\vec{r}\cdot\vec{r} - \frac{(\vec{r}\cdot\vec{a}_1)^2}{\vec{a}_1\cdot\vec{a}_1}\right)$$

$$\ell = \vec{r}\cdot\hat{a} = \frac{\vec{r}\cdot\vec{a}_1}{(\vec{a}_1\cdot\vec{a}_1)^{1/2}}$$

$$\hat{a} = \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$ = directional cosines (components of $\hat{a}$)

$$I_{OP} = \sum m'\left[(x^2+y^2+z^2) - (\ell x + my + nz)^2\right]$$
$$= \sum m'\left[(x^2+y^2+z^2)(\ell^2+m^2+n^2) - (\ell x + my + nz)^2\right]$$

$$I_{OP} = l^2 \sum m'(y^2+z^2) + m^2 \sum m'(x^2+z^2) + n^2 \sum m'(x^2+y^2) - 2lm \sum m'xy - 2ln \sum m'xz - 2mn \sum m'yz$$

$$I_{OP} = \underbrace{I_x l^2 + I_y m^2 + I_z n^2}_{\text{moments of inertia}} \underbrace{- 2I_{xy}lm - 2I_{xz}ln - 2I_{yz}mn}_{\text{products of inertia}}$$

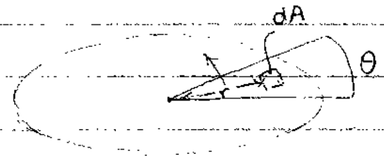
$$\therefore I_{OP} = [\,l \quad m \quad n\,] \underbrace{\begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix}}_{\text{moment of inertia matrix}} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

Defn: The ELLIPSOID OF INERTIA is the set of $[\,l \ m \ n\,]$ so that the expression $I_x l^2 + I_y m^2 + I_z n^2$ = constant (because could be that $I_x \neq I_y \neq I_z$).

Tutorial #5 10-11-05

6.6 Show that the generalized forces corresponding to x, y, θ are

$F_x = -Aa\dot{x}$ $F_y = -Aa\dot{y}$ $F_\theta = -\frac{1}{2}Aar^2\dot{\theta}$



$d\vec{F} = -a\,dA\,\vec{v}$

(viscous force proportional to velocity and opposite in direction)

$\vec{v} = \begin{bmatrix} \dot{x} - r\sin\theta\,\dot{\theta} \\ \dot{y} + r\cos\theta\,\dot{\theta} \end{bmatrix} \Rightarrow d\vec{F} = -a\,dA \begin{bmatrix} \dot{x} - r\sin\theta\,\dot{\theta} \\ \dot{y} + r\cos\theta\,\dot{\theta} \end{bmatrix}$ avg values of sinθ and cosθ are 0

$\therefore \vec{F}_{total} = -aA \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$? not $\dot{\theta}$ component? PITFALL

- the rotational motion does not affect motion in x and y directions
  ↳ it's just a TORQUE!

$dF_x = -a\,dA(\dot{x} - r\sin\theta\,\dot{\theta}) \Rightarrow F_x = \int dF_x = -aA\dot{x}$

$dF_y = -a\,dA(\dot{y} + r\cos\theta\,\dot{\theta}) \Rightarrow F_y = \int dF_y = -aA\dot{y}$

$dF_\theta = dF_x \frac{\partial x}{\partial\theta} + dF_y \frac{\partial y}{\partial\theta}$ $\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\}$ polar coordinates

$= (-a\,dA[\dot{x} - r\sin\theta\,\dot{\theta}])(-r\sin\theta) + (-a\,dA[\dot{y} + r\cos\theta\,\dot{\theta}])(r\cos\theta)$

$= -a\,dA[\dot{x}(-r\sin\theta) + \dot{y}(r\cos\theta)] - a\,dA[+r^2\sin^2\theta\,\dot{\theta} + r^2\cos^2\theta\,\dot{\theta}]$

$= -a\,dA[\dot{x}(-r\sin\theta) + \dot{y}(r\cos\theta)] - a\,dA\,r^2\dot{\theta}$

$F_\theta = -a\int dA\,(r^2\dot{\theta})$ $dA = 2\pi r\,dr$ since $A = \pi r^2$

$= -a\int(2\pi r\,dr)(r^2\dot{\theta})$

$= -2\pi a\dot{\theta}\int r^3 dr = -2\pi a\dot{\theta}\left(\frac{r^4}{4}\right) = -\frac{1}{2}\pi r^4 a\dot{\theta} = -\frac{1}{2}Aar^2\dot{\theta}$

Power function of a little part of the disk: (method 2)

$$dP = -\frac{1}{2} a v^2 \, dA = -\frac{1}{2} a \, dA \left[\dot{x}^2 + \dot{y}^2 + r^2\dot{\theta}^2 - 2r\dot{x}\sin\theta\,\dot{\theta} + 2r\dot{y}\cos\theta\,\dot{\theta} \right]$$

$$v^2 = v_x^2 + v_y^2 = (\dot{x} - r\sin\theta\,\dot{\theta})^2 + (\dot{y} + r\cos\theta\,\dot{\theta})^2$$

$$P = \int dP = -\frac{1}{2} aA(\dot{x}^2+\dot{y}^2) - \frac{1}{2} a \int r^2\dot{\theta}^2 \, 2\pi r \, dr \qquad dA = 2\pi r \, dr$$

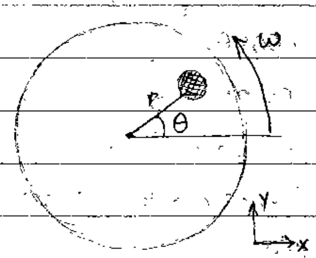
$$= -\frac{1}{2} aA(\dot{x}^2+\dot{y}^2) - \pi a \dot{\theta}^2 \left(\frac{r^4}{4}\right)$$

$$= -\frac{1}{2} aA(\dot{x}^2+\dot{y}^2) - \frac{1}{4} aAr^2\dot{\theta}^2$$

$$= -\frac{1}{2} aA\left(\dot{x}^2+\dot{y}^2 - \frac{1}{2} r^2\dot{\theta}^2\right)$$

$$F_x = \frac{\partial P}{\partial \dot{x}} = -Aa\dot{x} \qquad F_y = \frac{\partial P}{\partial \dot{y}} = -Aa\dot{y} \qquad F_\theta = \frac{\partial P}{\partial \dot{\theta}} = \frac{1}{2} Aar^2\dot{\theta}$$

6.10



a) Assume dry friction: $P = -av$ (v = speed, scalar)

$P = -a|\vec{v}|$

v = relative velocity of particle to disk

$= |\vec{v}_{part} - \vec{v}_{disk}|$

$$\vec{v}_{disk} = \begin{bmatrix} -r\omega\sin\theta \\ +r\omega\cos\theta \end{bmatrix} \qquad \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \qquad \therefore \vec{v}_{part} = \begin{bmatrix} \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \\ \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \end{bmatrix}$$

$$\Delta v = \begin{bmatrix} \dot{r}\cos\theta - r(\dot{\theta}-\omega)\sin\theta \\ \dot{r}\sin\theta + r(\dot{\theta}-\omega)\cos\theta \end{bmatrix}$$

$$(\Delta\vec{v})^2 = \dot{r}^2(\cos^2\theta + \sin^2\theta) + r^2(\dot{\theta}-\omega)^2(\sin^2\theta+\cos^2\theta)$$
$$-2r\dot{r}(\dot{\theta}-\omega)\sin\theta\cos\theta + 2r\dot{r}(\dot{\theta}-\omega)\sin\theta\cos\theta$$

$$(\Delta v)^2 = \dot{r}^2 + r^2(\dot{\theta}-\omega)^2$$

$$\therefore\ P = -a|\vec{v}| = -a\,(\dot r^2 + r^2(\dot\theta - \omega)^2)^{1/2}$$

$\uparrow$ $\boxed{\mu m g}$

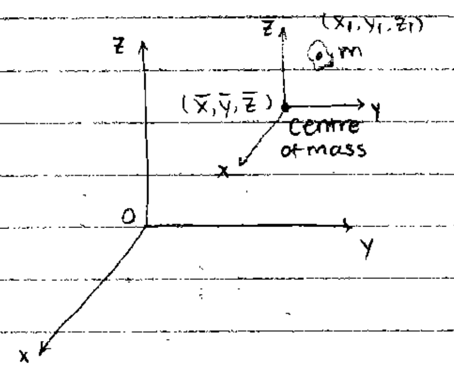
Ellipsoid of Inertia — 10-12-05

$$I_x x^2 + I_y y^2 + I_z z^2 - 2I_{xy}xy - 2I_{xz}xz - 2I_{yz}yz = 1$$

$$\begin{vmatrix} I_x - I_p & I_{xy} & I_{xz} \\ I_{xy} & I_y - I_p & I_{xz} \\ I_{xz} & I_{yz} & I_z - I_p \end{vmatrix} = 0$$

THREE solutions for I_p (eigenvalues) give moment of inertia about principle axes (of ellipsoid)

Same as diagonalizing: $A = S^{-1}\Lambda S$



$(\bar x, \bar y, \bar z)$ coordinates of the CM in XYZ (rel. to O)

(x_1, y_1, z_1) coords of m rel. to $(\bar x, \bar y, \bar z)$

$$I_z = \sum m(x^2 + y^2)$$
$$= \sum m((x_1+\bar x)^2 + (y_1 + \bar y)^2)$$
$$= \sum m(x_1^2 + y_1^2) + \sum m(\bar x^2 + \bar y^2) + 2\bar x \underbrace{\sum m x_1}_{0} + 2\bar y \underbrace{\sum m y_1}_{0}$$

$$\therefore\ I_z = \bar I_z + M(\bar x^2 + \bar y^2)$$
$$I_{xy} = \bar I_{xy} + M\bar x\bar y$$
etc.

moment of inertia of particle = moment of inertia about some point + moment of inertia of that point about CM

$$I_{xy} = \sum mxy = \sum m(\bar x + x_1)(\bar y + y_1) = \sum m\bar x\bar y + \sum m x_1\bar y + \sum m\bar x y_1 + \sum m x_1 y_1$$
$$= \bar I_{xy} + M\bar x\bar y$$

$$[\bar I_x + M(\bar y^2 + \bar z^2)]x^2 + [\bar I_y + M(\bar x^2 + \bar z^2)]y^2 + [\bar I_z + M(\bar x^2 + \bar y^2)]z^2$$
$$- 2[\bar I_{xy} + M\bar x\bar y]xy - 2[\bar I_{xz} + M\bar x\bar z]xz - 2[\bar I_{yz} + M\bar y\bar z]yz = 0$$

Pick $\bar x, \bar y, \bar z$ to be principal axes about the CM, then $\bar I_{xy} = \bar I_{xz} = \bar I_{yz} = 0$

If at centre of mass: $\bar{x} = \bar{y} = \bar{z} = 0$

If along principal axes: $\bar{x} = \bar{y} = 0$ if along z

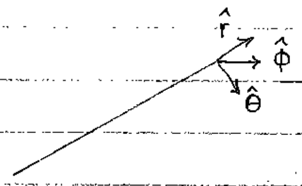
$\bar{x} = \bar{z} = 0$ along y

$\bar{y} = \bar{z} = 0$ along x

Conclusion:

THE PRINCIPAL AXES ALONG A PRINCIPAL AXES RELATIVE TO CENTRE OF MASS COINCIDE WITH THOSE IN CM

From Homework #4 10-14-05



$$\vec{F} = -a(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi})$$

$$F_\phi = (\vec{F}\cdot\hat{\phi})\frac{\partial l_{\hat{\phi}}}{\partial\phi} + (\vec{F}\cdot\hat{\theta})\frac{\partial l_{\hat{\theta}}}{\partial\phi} + (\vec{F}\cdot\hat{r})\frac{\partial l_{\hat{r}}}{\partial\phi}$$

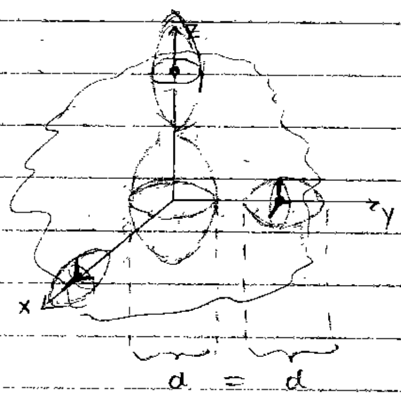
$$= \frac{\partial l_{\hat{\phi}}}{\partial\phi}(\vec{F}\cdot\hat{\phi})$$

$$= r\sin\theta\,(\vec{F}\cdot\hat{\phi}) \qquad \left(\frac{\partial l_{\hat{\phi}}}{\partial\phi} = r\sin\theta\right)$$

$$F_\phi = F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} + F_z\frac{\partial z}{\partial\phi} \qquad F_x = -a\left(\frac{\partial x}{\partial r}\dot{r} + \frac{\partial x}{\partial\theta}\dot{\theta} + \frac{\partial x}{\partial\phi}\dot{\phi}\right)$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$$

Moments of inertia

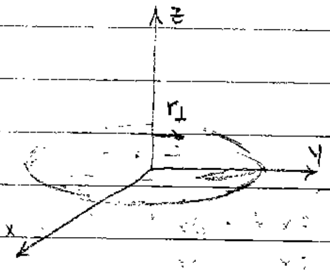


- as you move along an axis (say y), the moment of inertia along that axis stays constant (the axis of the ellipsoids' along y-direction are the same length, while the axes in the x and z directions can change in length

- ellipsoid of inertia of an ellipsoid

↳ Equation for an ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(cf. inertia ellipsoid $I_x x^2 + I_y y^2 + I_z z^2 = 1$)



$$I_z = \int_V \rho r_\perp^2 \, dV = \int_V \rho (x^2+y^2) \, dV$$

Let $x' = \frac{x}{a}$, $y' = \frac{y}{b}$, $z' = \frac{z}{c}$ $\quad x'^2 + y'^2 + z'^2 = 1$

$$\text{Jacobian} = \left| \frac{\partial(x,y,z)}{\partial(x',y',z')} \right| = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{vmatrix} = \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$$

$$I_z = \int_{V'} \rho \left[(ax')^2 + (by')^2 \right] dx'dy'dz' \left| \frac{\partial(x,y,z)}{\partial(x',y',z')} \right|$$

$$= abc\rho \int_0^1 \int_0^{2\pi} \int_0^{\pi} (a^2 r^2 \cos^2\phi \sin^2\theta + b^2 r^2 \sin^2\phi \sin^2\theta) \, r \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{4\pi \rho abc}{15} (a^2+b^2+c^2) = \frac{1}{5} M (a^2+b^2)$$

since $V = \frac{4}{3}\pi abc$

$\therefore M = \rho V = \rho \frac{4\pi abc}{3}$

$\therefore I_x = \frac{1}{5} M (b^2+c^2)$

$I_y = \frac{1}{5} M (a^2+c^2)$

Take $M = 5$:

$\therefore (b^2+c^2) x^2 + (a^2+c^2) y^2 + (a^2+b^2) z^2 = 1$

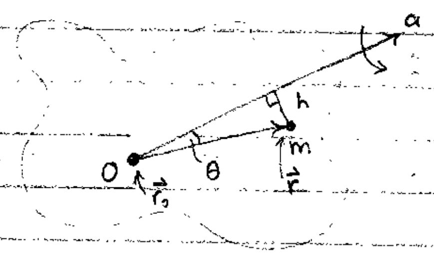
Rigid Body

10-17-05

- no part of the body moves relative to another
- a rigid body is equivalent to a system of particles (w/ constraint)

- Two Methods:
  a) Lagrangian: write out T ⇒ $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$
  b) Euler method: ⇒ Euler's equations

- ANGULAR VELOCITY:



- velocity of m is perpendicular to the plane defined by Oam and magnitude is ωh (circular path around axis Oa)
- $\vec{Oa} \times \vec{Om}$ points into the page, is perpendicular to the plane Oam in the direction of $\vec{v}$

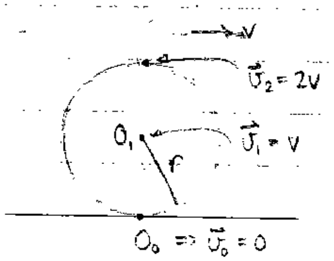
$$|\vec{Oa} \times \vec{Om}| = |\vec{Oa}||\vec{Om}|\sin\theta \qquad |\vec{Om}|\sin\theta = h$$
$$= |\vec{Oa}|\,h$$

- define $\vec{\omega} \parallel \vec{Oa}$, $|\vec{\omega}| = \omega$ $\quad \boxed{\therefore\ \vec{\omega} \times \vec{Om} = \vec{v}_m} \quad (\vec{\omega} \times \vec{r})$

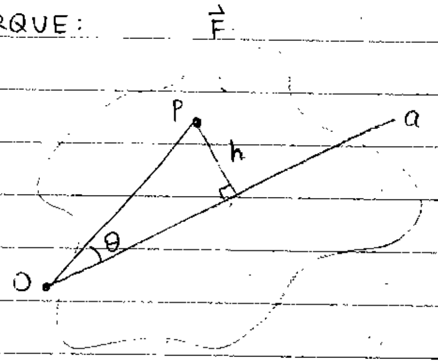
$$\vec{\omega}_T = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3 + \ldots$$

- for object which translates and rotates:
$$\vec{v} = \vec{v}_0 + \vec{\omega} \times (\vec{r} - \vec{r}_0)$$

- if we move the origin:
$$\vec{v} = \vec{v}_0 + \vec{\omega} \times (\vec{r} - \vec{r}_0) = \vec{v}_1 + \vec{\omega} \times (\vec{r} - \vec{r}_1)$$
$$\therefore\ \vec{v}_1 = \vec{v}_0 + \vec{\omega} \times (\vec{r}_1 - \vec{r}_0)$$
→ angular velocity is the same, as origin moves around



- TORQUE:



- Torque about $\vec{Oa}$ is the force normal to the Oap plane times h.

$h = |\vec{Op}| \sin\theta$

$\tau_{oa} = \frac{[\vec{Oa} \times \vec{Op}] \cdot \vec{F}}{|\vec{Oa}|}$

$|\vec{Oa} \times \vec{Op}| = |\vec{Oa}||\vec{Op}| \sin\theta = |\vec{Oa}| h$

(numerator gives a vector ⊥ Oap)

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

$\therefore \tau_{oa} = \frac{\vec{Oa} \cdot [\vec{Op} \times \vec{F}]}{|\vec{Oa}|} = \frac{\vec{Oa}}{|\vec{Oa}|}[\vec{Op} \times \vec{F}] = \frac{\vec{Oa}}{|\vec{Oa}|} \cdot \vec{\tau}$ $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$

= projection of $\vec{\tau}$ onto $\vec{Oa}$

- Kinetic energy of a rigid body?

$T = \frac{1}{2} \sum m (v_x^2 + v_y^2 + v_z^2)$ $\vec{v} = \vec{v}_o + \vec{\omega} \times \vec{r} \Rightarrow v_x = v_{ox} + \omega_y z - \omega_z y$

$v_y = v_{oy} + \omega_z x - \omega_x z$

$v_z = v_{oz} + \omega_x y - \omega_y x$

$v_x^2 = v_{ox}^2 + \omega_y^2 z^2 + \omega_z^2 y^2 + 2v_{ox}\omega_y z - 2v_{ox}\omega_z y - 2\omega_y\omega_z yz$

$v_y^2 = v_{oy}^2 + \omega_z^2 x^2 + \omega_x^2 z^2 + 2v_{oy}\omega_z x - 2v_{oy}\omega_x z - 2\omega_x\omega_z xz$

$v_z^2 = v_{oz}^2 + \omega_x^2 y^2 + \omega_y^2 x^2 + 2v_{oz}\omega_x y - 2v_{oz}\omega_y x - 2\omega_x\omega_y xy$

$$\therefore T = \frac{1}{2} M v_o^2 + \frac{1}{2}\omega_x^2 \overbrace{\sum m(y^2+z^2)}^{I_x} + \frac{1}{2}\omega_y^2 \overbrace{\sum m(x^2+z^2)}^{I_y} + \frac{1}{2}\omega_z^2 \overbrace{\sum m(x^2+y^2)}^{I_z}$$

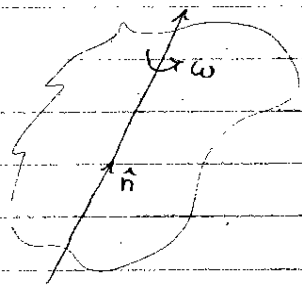
$$+ M v_{ox}(\omega_y \bar{z} - \omega_z \bar{y}) + M v_{oy}(\omega_z \bar{x} - \omega_x \bar{z}) + M v_{oz}(\omega_x \bar{y} - \omega_y \bar{x})$$

$$+ \omega_y \omega_z \underbrace{\sum myz}_{I_{yz}} + \omega_x \omega_z \underbrace{\sum mxz}_{I_{xz}} + \omega_x \omega_y \underbrace{\sum mxy}_{I_{xy}}$$

$$T = \frac{1}{2} M v_o^2 + \frac{1}{2} \vec{\omega} \cdot (\bar{\bar{I}} \vec{\omega}) + M \vec{v}_o \cdot (\vec{\omega} \times \vec{r}_{cm})$$

$\bar{\bar{I}}$ is inertia matrix

Tutorial #6



$\vec{\omega} = \omega\hat{n}$ (instantaneous axis of rotation)

$\rho(\vec{x})$ = density (distribution of mass) $= \frac{M}{V}$ (if uniform)

$$I_{ij} = \int_V \rho(\vec{x})\,(\delta_{ij}\vec{x}^2 - x_i x_j)\,dV$$

$i,j = x,y,z$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$(x_x, x_y, x_z) = \vec{x}$

$$I_{xx} = \int_V \rho(\vec{x})\,(\delta_{xx}\vec{x}^2 - x_x x_x)\,dV$$

I_{xx}, I_{yy}, I_{zz} = moments of inertia

$$= \int_V \rho(\vec{x})\,(x^2+y^2+z^2-x^2)\,dV$$

$$= \int_V \rho(\vec{x})\,(y^2+z^2)\,dV$$

$$I_{xy} = \int_V \rho(\vec{x})\,(\delta_{xy}\vec{x}^2 - x_x x_y)\,dV$$

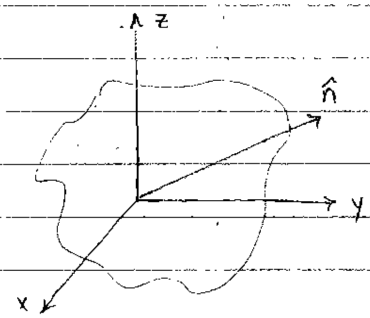
I_{xy}, I_{yz}, I_{xz} = products of inertia

$$= \int_V \rho(\vec{x})\,(0 - xy)\,dV$$

$$= -\int_V \rho(\vec{x})\,xy\,dV$$

momentum $= L_i = \sum_j I_{ij}\omega_j$

kinetic energy $= T = \frac{1}{2}\sum_{i,j} I_{ij}\omega_i\omega_j$



$$I_{\hat{n}} = \sum_{i,j} I_{ij} \hat{n}_i \hat{n}_j$$

$$= \sum_i I_{ix} n_i n_x + \sum_i I_{iy} n_i n_y + \sum_i I_{iz} n_i n_z$$

$\hat{n} = (n_x, n_y, n_z)$

$\hat{n} \cdot \hat{n} = 1$

- Remarks:
  - ↳ I_{ij} is constant but coordinate dependent
  - ↳ $\vec{\omega}$ is not constant in general

- $\delta_{ij} = \delta_{ji}$; $x_i x_j = x_j x_i$; $I_{ij} = I_{ji}$ ($I_{xy} = I_{yx}$, ... ⇒ I is symmetric matrix)

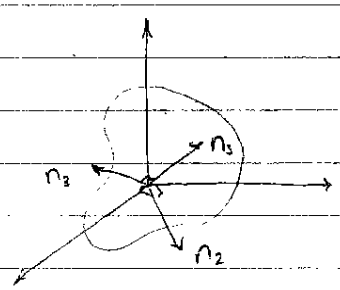
$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

- Principle Moments:

  $\{ I_1, I_2, I_3 \}$ = Eigenvalues (I)

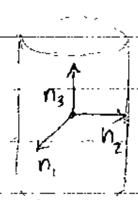
- Principal Axes:

  $\{ \hat{n}_1, \hat{n}_2, \hat{n}_3 \}$ = Eigenvectors (I)



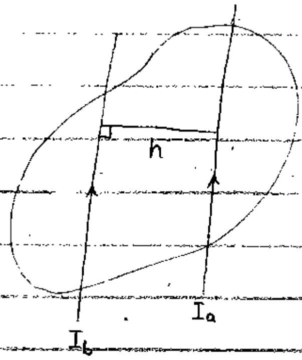
$$I^{(n)} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

n_1, n_2, n_3 ⊥ basis



most symmetric about n_1, n_2, n_3

- Parallel Axis Theorem:



$I_b = I_a + Mh^2$

#1. $I_{0a} = \frac{1}{(x^2+y^2+z^2)}(I_x x^2 + I_y y^2 + I_z z^2 \pm 2I_{xy}xy \pm 2I_{yz}yz \pm 2I_{xz}xz)$

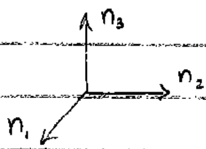
Ex. Let $\hat{n} = \frac{1}{\sqrt{2}}(0,1,1)$, moment of inertia about $\hat{n}$ as axis:

$I_{\hat{n}} = \sum_i I_{ix}n_in_x + \sum_i I_{iy}n_in_y + \sum_i I_{iz}n_in_z$

$= 0 + (I_{xy}n_xn_y + I_{yy}n_yn_y + I_{zy}n_zn_y) + (I_{xz}n_xn_z + I_{yz}n_yn_z + I_{zz}n_zn_z)$

$$I_{\hat{n}} = (n_x\ n_y\ n_z)\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix} n_x \\ n_y \\ n_z\end{pmatrix}$$

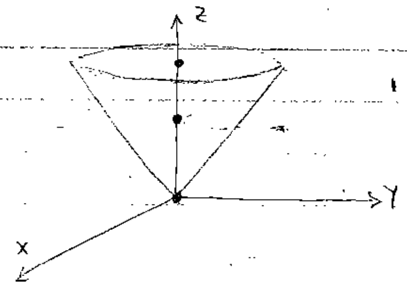
- Inertial Ellipsoid:



ellipsoid with axis in directions n_1, n_2, n_3 and lengths (diameters) in those directions as I_1, I_2, I_3

↳ geometric representation on the symmetry of an object

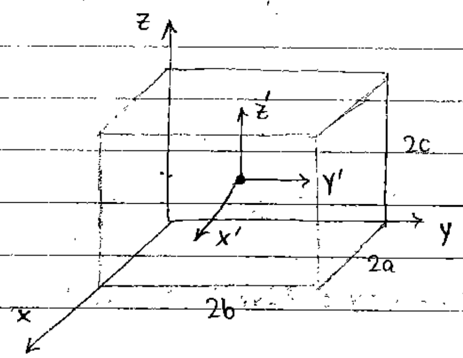
#2.



Calculate I_{ij}

#3. Given any object, prove that a system of 3 rods can be constructed to have the same principal moments.

- Example 7.4 (Wells)



$\rho(x) = \frac{M}{V} = \frac{M}{(2a)(2b)(2c)} = \frac{M}{8abc}$ (constant)

(homogeneous box).

$I_{x'} = \int_V \rho(\vec{x})\,(y^2+z^2)\,dV$

$= \frac{M}{8abc}\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c}(y^2+z^2)\,dx'dy'dz'$

$I_{x'} = \frac{M}{8abc}\left[\left(\int_{-a}^{a}dx'\right)\left(\int_{-c}^{c}dz'\right)\left(\int_{-b}^{b}y^2dy'\right) + \left(\int_{-a}^{a}dx'\right)\left(\int_{-b}^{b}dy\right)\left(\int_{-c}^{c}z^2dz'\right)\right]$

$= \frac{M}{8abc}\left[(2a)(2c)\left(\frac{1}{3}y^3\Big|_{-b}^{b}\right) + (2a)(2b)\left(\frac{1}{3}z^3\Big|_{-c}^{c}\right)\right]$

$= \frac{M}{8abc}\left[\frac{4ac}{3}(b^3+b^3) + \frac{4ab}{3}(c^3+c^3)\right]$

$= \frac{M}{8abc}\left[\frac{8acb^3}{3} + \frac{8abc^3}{3}\right] = \frac{M}{3}(b^3+c^3)$

$I_{y'} = \int_V \rho(\vec{x})\,(x'^2+z'^2)\,dV = \frac{M}{3}(a^3+c^3)$ $\quad I_{z'} = \frac{M}{3}($

10-19-05

$$T = \underbrace{\frac{1}{2} M v_0^2}_{\text{origin of coords moving}} + \underbrace{\frac{1}{2} \vec{\omega} \cdot (\vec{\vec{I}} \vec{\omega})}_{\text{moment of inertia tensor}} + \underbrace{M\vec{v}_0 \cdot (\vec{\omega} \times \vec{r}_{CM})}_{\text{motion of centre of mass}}$$

Kinetic energy about the centre of mass: $T_{cm} = \frac{1}{2} \vec{\omega} \cdot (\vec{\vec{I}} \vec{\omega})$ constant (internal forces do no work)

(since motion = (motion of CM) + (motion around CM))

$\vec{L} = \sum m \vec{r} \times \vec{v}$ $\quad \vec{v} = \vec{v}_0 + \vec{\omega} \times \vec{r}$ $\quad \vec{L}$ = angular momentum

$$\therefore \vec{r} \times \vec{v} = \vec{r} \times \vec{v}_0 + \vec{r} \times (\vec{\omega} \times \vec{r})$$

Vector Triple Product: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\therefore \vec{r} \times \vec{v} = \vec{r} \times \vec{v}_0 + [\vec{\omega}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot \vec{\omega})] = \vec{r} \times \vec{v}_0 + \vec{\omega} r^2 - \vec{r}(\vec{r} \cdot \vec{\omega})$$

i.e. $\vec{L}$ is not in same direction as $\vec{\omega}$!

$$\vec{L} = \sum m \vec{r} \times \vec{v} = \sum m (\vec{r} \times \vec{v}_0 + \vec{\omega} r^2 - \vec{r}(\vec{\omega} \cdot \vec{r}))$$

$$= M\vec{r}_{cm} \times \vec{v}_0 + \sum m \left(\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} (x^2 + y^2 + z^2) - \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{r}} \underbrace{(\omega_x x + \omega_y y + \omega_z z)}_{\vec{\omega} \cdot \vec{r}} \right)$$

$$= M\vec{r}_{cm} \times \vec{v}_0 + \sum m \begin{bmatrix} \omega_x (y^2 + z^2) - \omega_y xy - \omega_z xz \\ \omega_y (x^2 + z^2) - \omega_x xy - \omega_z yz \\ \omega_z (x^2 + y^2) - \omega_x xz - \omega_y yz \end{bmatrix}$$

$$\therefore \vec{L} = M\vec{r}_{cm} \times \vec{v}_0 + \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$ constant

Take $\vec{v}_0 = 0 \Rightarrow \vec{L} = \vec{\vec{I}} \vec{\omega}$

$$T_{cm} = \frac{1}{2} \vec{\omega} \cdot (\vec{\vec{I}} \omega) = \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

- If $\vec{L}$ is conserved (no torques) and T is conserved, is $\vec{\omega}$ conserved?
  - ↳ $\vec{\omega}$ not conserved (motion is not only rotation around the axis of the object, but also some tumbling motion)
  - ↳ magnitude of $\vec{\omega}$ may be conserved, but changes direction (mostly)

$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}|\vec{\omega}||\vec{L}|\cos\theta$ θ = angle between $\vec{\omega}$ and $\vec{L}$

$\vec{L}$ is conserved $\Rightarrow$ $|\vec{L}|$ is constant

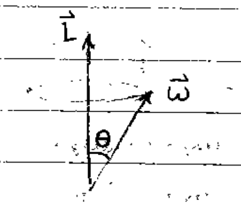
* $\vec{\omega}$ would be conserved if $\vec{L} \parallel \vec{\omega}$ $\vec{L} = \overleftrightarrow{I}\vec{\omega}$

↳ means that $\vec{\omega}$ is an eigenvector of $\overleftrightarrow{I}$

i.e. $\overleftrightarrow{I}\vec{\omega} = \lambda\vec{\omega} = \vec{L}$, $\vec{L} \parallel \vec{\omega}$ (λ eigenvalue)

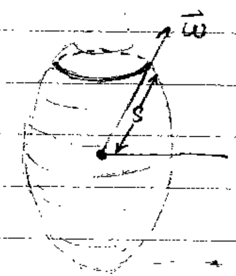
- In general, $\vec{\omega}$ changes even when there are no torques.

Ex for a basketball thrown, moments of Inertia are about equal in all 3 directions ($I_x = I_y = I_z$), $\overleftrightarrow{I}$ reduces to an identity matrix ∴ all vectors are eigenvectors of $\overleftrightarrow{I}$



$\vec{\omega}$ precesses about $\vec{L}$ with a constant angle θ

- Inertia ellipsoid $I^P_{xx} = I^P_{yy}$



- $\vec{\omega}$ traces out a circle around the vertical axis through the object ⇒ "punctures" out a circular path on the object
  - ↳ the "North pole" → changes location on the object as angle between $\vec{\omega}$ and $\vec{L}$ (vertical)

- For a two-dimensional object: $I_x = \sum m(y^2+\cancel{z^2}) = \sum m(y^2)$

$$I_y = \sum m(x^2+\cancel{z^2}) = \sum m(x^2)$$

$$I_z = \sum m(x^2+y^2)$$

$\therefore\ I_z = I_x + I_y$

- for HW#5 cone question: $I_z = \frac{1}{2}MR^2$ (for disks)

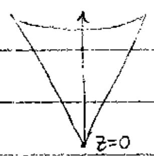
since $I_x = I_y$, $I_x = I_y = \frac{1}{4}MR^2$

$$dI_z = \frac{1}{2}(\pi r^2)\rho\, dz\, r^2 \qquad dI_x = \left[\frac{1}{4}(\pi r^2)r^2 + \pi r^2 z^2\right]\rho\, dz$$

(from Parallel Axis Theorem)

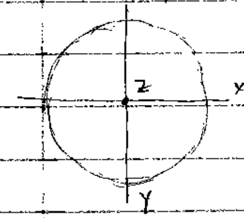
- For a cone shell: $dI_z = Mr^2 = 2\pi r\, t\rho\, dz\, r^2$ $\quad$ t = thickness

(moment of inertia of a ring)



$$r = \frac{R}{h}z \qquad \int dI_z = \int_0^h 2\pi\left(\frac{R}{h}\right)z\, t\rho\, dz\left(\left(\frac{R}{h}\right)z\right)^2$$

$$= 2\pi\left(\frac{R}{h}\right)^3 t\rho\, \frac{h^4}{4}$$



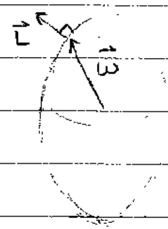
$$M = \int dm = \int_0^h 2\pi\left(\frac{R}{h}\right)z\, t\rho\, dz = \pi\left(\frac{R}{h}\right)h^2 t\rho$$

$$\therefore\ I_z = \frac{2\pi\left(\frac{R}{h}\right)^3 t\rho\frac{h^4}{4}}{\pi\left(\frac{R}{h}\right)t\rho h^2} =$$

$$dI_x = \frac{1}{2}MR^2 + Mz^2$$

$$\int dI_x = \frac{1}{4}MR^2 + \int_0^h 2\pi r t\rho\, dz\, z^2 = \frac{1}{4}MR^2 + 2\pi\left(\frac{R}{h}\right)t\rho\frac{h^4}{4}$$

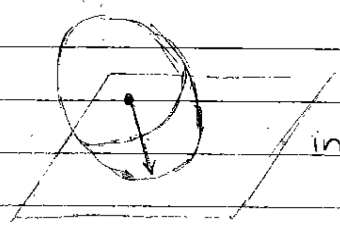
$$= \frac{1}{4}MR^2 + \frac{1}{2}Mh^2$$



$\vec{\omega}$ confined to sit on an ellipsoid

$T = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\vec{\omega}$ $\quad \therefore \nabla T_\omega = \overleftrightarrow{I}\vec{\omega} = \vec{L}$

$\therefore$ as T increases, it increases in the direction of its gradient $\nabla T_\omega = \vec{L}$ so that a larger ellipsoid is created $\rightarrow$ $\vec{L}$ is $\perp$ to surface of ellipsoid.



invariable plane $\rightarrow$ plane $\perp$ $\vec{L}$ (angular momentum)
↳ ellipsoid tumbles/rolls on the planes, never penetrates

$$2T = \omega_x^2 I_x^P + \omega_y^2 I_y^P + \omega_z^2 I_z^P$$

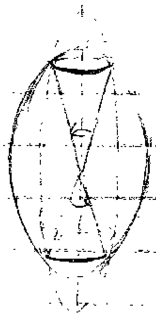
$$\vec{L} = \omega_x I_x^P \hat{x} + \omega_y I_y^P \hat{y} + \omega_z I_z \hat{z}$$
$$|\vec{L}|^2 = \omega_x^2 (I_x^P)^2 + \omega_y^2 (I_y^P)^2 + \omega_z (I_z^P)^2$$

$\vec{\omega}$ restricted to a curve which is the intersection of the inertia ellipsoid and a second ellipsoid

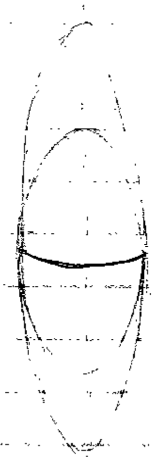
$$= \frac{\omega_x^2 (I_x^P)^2}{\left(\frac{L^2}{2T}\right)} + \frac{\omega_y^2 (I_y^P)^2}{\left(\frac{L^2}{2T}\right)} + \frac{\omega_z^2 (I_z^P)^2}{\left(\frac{L^2}{2T}\right)} = \frac{|\vec{L}|^2 (2T)}{L^2} = 2T$$

Intersection: $\omega_x^2 I_x^P + \omega_y^2 I_y^P + \omega_z^2 I_z^P = \frac{\omega_x^2 (I_x^P)^2}{\left(\frac{L^2}{2T}\right)} + \frac{\omega_y^2 (I_y^P)^2}{\left(\frac{L^2}{2T}\right)} + \frac{\omega_z (I_z^P)^2}{\left(\frac{L^2}{2T}\right)}$

eg if $I_x = 1$, $I_y = 2$, $I_z = 3$, we can see that the 2nd ellipsoid is always more ellipsoidal (squshed)



$\vec{\omega}$ constrained to the 2 circles

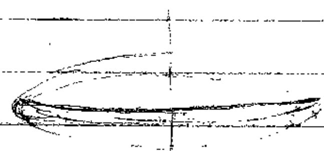


maximized $\vec{L}$ at a given T ($\vec{\omega}$ has full range)

maximized T at given $\vec{L}$ ($\vec{\omega}$ restricted to 2 points)

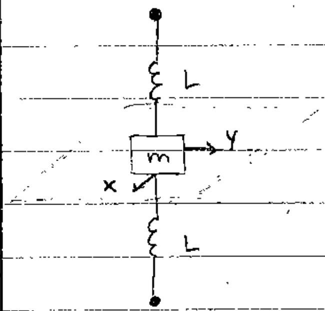
not possible to have so much $\vec{L}$ for given T

for frisbee



Midterm Solutions 10-25-05

1.



$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2$$

$$V = \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2 = k\left[\sqrt{x^2+y^2+L^2} - \ell\right]^2$$

$$\therefore L = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 - k\left[\sqrt{x^2+y^2+L^2} - \ell\right]^2$$

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} - L = T + V = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 + k\left[\sqrt{x^2+y^2+L^2} - \ell\right]^2$$

$$L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r\dot{\theta}^2 - k\left[\sqrt{r^2+L^2} - \ell\right]^2 \Rightarrow \frac{\partial L}{\partial \theta} = 0$$

$\frac{\partial L}{\partial \dot{\theta}}$ is conserved $\quad \frac{\partial L}{\partial \dot{\theta}} = m r\dot{\theta}$ (angular memomentum)

$$m\ddot{x} + 2k\left[\sqrt{x^2+y^2+L^2} - \ell\right] \frac{x}{\sqrt{x^2+y^2+L^2}} = 0$$

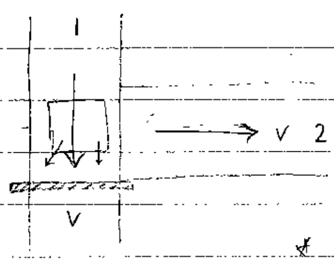
$$m\ddot{y} + 2k\left[\sqrt{x^2+y^2+L^2} - \ell\right] \frac{y}{\sqrt{x^2+y^2+L^2}} = 0$$

Small oscillations $\Rightarrow x^2, y^2 \ll L^2$ $\quad \ddot{x} = -\frac{2k}{m}[L-\ell]\frac{x}{L}$

$$\ddot{y} = -\frac{2k}{m}[L-\ell]\frac{y}{L}$$

$\therefore \omega_x = \omega_y = \sqrt{\frac{2k}{m}\frac{L-\ell}{L}}$ The trajectory is an ellipse.
(b/c general soln is like $A\cos\omega t + B\sin\omega t$)

2.



$$P_1 = -a_1 v = -a_1(\dot{x}^2 + v^2)^{1/2}$$

$$F_x^1 = \frac{\partial P}{\partial \dot{x}} = -a_1 \frac{\dot{x}}{\sqrt{\dot{x}^2+v^2}} \qquad F_x^2 = a_2$$

* only don't have constant when package is travelling in a different direction than belt

$$\therefore\ a_1 \frac{\dot{x}}{\sqrt{\dot{x}^2+v^2}} = a_2 \qquad a_1, a_2 \propto \text{area on that belt}$$

when on 2nd belt, $\dot{x}$ wrt belt 1 is just v

$$\therefore\ a_1 \frac{v}{\sqrt{v^2+v^2}} = a_2 \Rightarrow a_1\left(\frac{1}{\sqrt{2}}\right) = a_2$$

$$\therefore\ \frac{a_2}{a_1+a_2} = \frac{a_1\left(\frac{1}{\sqrt{2}}\right)}{a_1+a_1\left(\frac{1}{\sqrt{2}}\right)} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1}\frac{(\sqrt{2}-1)}{(\sqrt{2}-1)} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$$

Rigid Bodies

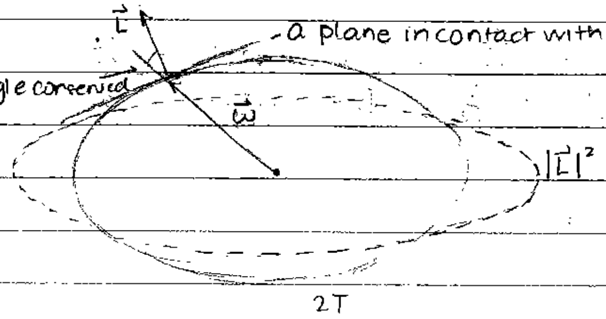
10-26-05

- angular momentum ($\vec{L}$) and kinetic energy (T) conserved

  $\vec{L} = I\vec{\omega}$

  $T = \frac{1}{2}\vec{\omega}\cdot(I\vec{\omega})$ I = moment of inertia tensor

- Ellipsoid of values of $\vec{\omega}$ that satisfies expression for T



$T = \frac{1}{2}\vec{L}\cdot\vec{\omega} = \frac{1}{2}|\vec{L}||\vec{\omega}|\cos\theta$

↳ $\theta, \vec{L}$ constant

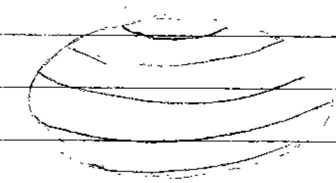
- 3 principle axes: 1, 2, 3

  $|\vec{L}|^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$
  $2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$

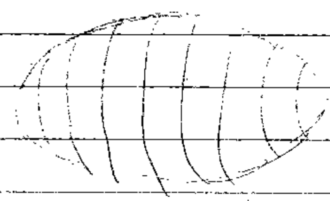
  } Both are conserved ∴ $\vec{\omega}$ must live in the intersection of these ellipsoids

  - ellipsoid corresponding to $\vec{L}$ is always more "squished" (higher aspect ratio due to I_1^2, I_2^2, I_3^2 terms)

- prolate spheroid: - oblate spheroid



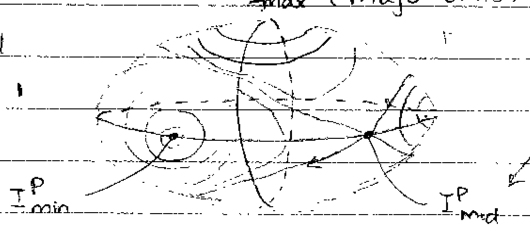
means that 2 of the I's are =



($\vec{\omega}$ lives along these lines, which are circles)

- For a triaxial ellipsoid (none are =)

  curves of intersection not necessarily ellipsoids (for 2 triaxial ellipsoids)



$\frac{d\vec{L}}{dt}$ = rate of change of angular momentum = torques = $\vec{N}$

$$\vec{L} = L_1\hat{1} + L_2\hat{2} + L_3\hat{3} \Rightarrow \frac{d\vec{L}}{dt} = \frac{\partial L_1}{\partial t}\hat{1} + \frac{\partial L_2}{\partial t}\hat{2} + \frac{\partial L_3}{\partial t}\hat{3} + L_1\frac{\partial \hat{1}}{\partial t} + L_2\frac{\partial \hat{2}}{\partial t} + L_3\frac{\partial \hat{3}}{\partial t}$$

- If body in rigid rotation,

$\odot$ $\vec{\omega}$ pointing out of page

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Right Hand Rule

$$\frac{d\vec{L}}{dt} = \frac{\partial L_1}{\partial t}\hat{1} + \frac{\partial L_2}{\partial t}\hat{2} + \frac{\partial L_3}{\partial t}\hat{3} + L_1\vec{\omega}\times\hat{1} + L_2\vec{\omega}\times\hat{2} + L_3\vec{\omega}\times\hat{3}$$

Define $\vec{L}_B = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$ $\quad \therefore \frac{d\vec{L}}{dt} = \frac{d\vec{L}_B}{dt} + \vec{\omega}\times\vec{L}_B = \vec{N}$

(note $\vec{L}_B = L_1\hat{1} + L_2\hat{2} + L_3\hat{3}$)

$I_i\dot{\omega}_i + \epsilon_{ijk}\,\omega_j\omega_k I_k = N_i$

pick all sets of ijk

eg. if $i=1$, $j=2$ & $k=3$

OR $j=3$ & $k=2$

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

- If $\vec{N}$ (torque) vanishes:

$$\left.\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) \\ I_2\dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) \end{aligned}\right\} \text{Euler equation:}$$

- So if $\vec{\omega}$ is constant, then $\vec{\omega}$ // PA

Notation : $(\vec{x}\times\vec{y})_i = \epsilon_{ijk}x_j y_k \Rightarrow$ eg. $(\vec{x}\times\vec{y})_2 = x_1y_3 - x_3y_1$

For $I_1 = I_2$ (spheroid):

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_1 - I_3)$$

$$I_1 \dot{\omega}_2 = -\omega_3 \omega_1 (I_1 - I_3)$$

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) = 0 \quad \Rightarrow$$ ω_3 corresponding to the axis that is unique does not change with time

Observe: $\dot{\omega}_1 = -\Omega \omega_2$ $\qquad \Omega = \frac{I_3 - I_1}{I_1} \omega_3$

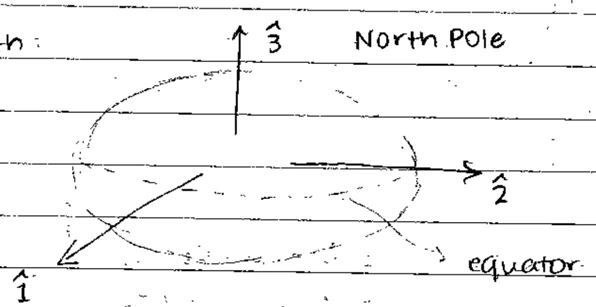
$\dot{\omega}_2 = \Omega \omega_1$

$$\omega_1 = A \cos \Omega t$$

$$\omega_2 = A \sin \Omega t$$

$$\omega_3 = \text{constant} = \left(\frac{I_1}{I_3 - I_1} \right) \Omega$$

The Earth:



$$\frac{I_1}{I_3 - I_1} = \frac{R_{eq}^2 + R_p^2}{2R_{eq}^2 - (R_{eq}^2 + R_p^2)} = \frac{R_{eq}^2 + R_p^2}{R_{eq}^2 - R_p^2}$$

$$\approx \frac{2R_{eq}^2}{(R_{eq} + R_p)(R_{eq} - R_p)}$$

$$= \frac{R_{eq}}{R_{eq} - R_p} = \frac{6300}{22} = 300$$

Euler Angles 10-28-05

$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_3\omega_2$ Take $I_3 > I_2 > I_1$ $\dot{\omega}_1 = -A\omega_3\omega_2$

$I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$ $\dot{\omega}_2 = +B\omega_3\omega_1$

$I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$ $\dot{\omega}_3 = -C\omega_1\omega_2$

Where $A = \dfrac{I_3 - I_2}{I_1}$, $B = \dfrac{I_3 - I_1}{I_2}$, $C = \dfrac{I_2 - I_1}{I_3}$

Take $\omega_1 \gg \dot{\omega}_2, \omega_3$ $\dot{\omega}_1 \approx 0$

$\dot{\omega}_2 = +B\omega_3\omega_1$ $\dot{\omega}_2 = \sqrt{BC}\,\omega_3'\omega_1$

$\dot{\omega}_3 = -C\omega_1\omega_2$ $\dot{\omega}_3' = -\sqrt{BC}\,\omega_1\omega_2$

let $\omega_3 = \sqrt{\dfrac{C}{B}}\,\omega_3'$ Let $\Omega = \sqrt{BC}\,\omega_1$

$(\alpha^2 = -BC\omega_1^2,\ \Omega = i\alpha)$

$\therefore \omega_3 = \sqrt{\dfrac{C}{B}}\,K\cos\Omega t$ $\omega_2 = K\sin\Omega t$

Take $\omega_3 \gg \omega_1, \omega_2$ $\dot{\omega}_3 \approx 0$ Let $\Omega = \sqrt{AB}\,\omega_3$

$\dot{\omega}_1 = -A\omega_3\omega_2$ $\therefore \omega_1 = \sqrt{\dfrac{A}{B}}\,K\cos\Omega t$

$\dot{\omega}_2 = +B\omega_3\omega_1$

$\omega_2 = K\sin\Omega t$

Take $\omega_2 \gg \omega_3, \omega_1$ $\dot{\omega}_2 \approx 0$

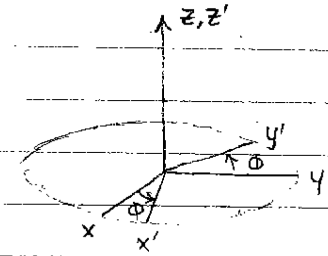
$\dot{\omega}_1 = -A\omega_3\omega_2$ $\omega_1 =$

$\dot{\omega}_3 = -C\omega_1\omega_2$

$\alpha\omega_1 = -A\omega_3\omega_2$ $\alpha\omega_3 = -C\omega_1\omega_2$

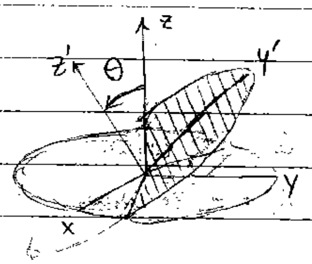
$= A\left(\dfrac{C\omega_1\omega_2}{\alpha}\right)\omega_2$

$\therefore \alpha = \pm\sqrt{AC}\,\omega_2$ $\alpha^2 = AC\omega_2^2$



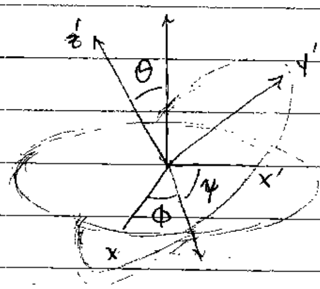
rotation about z' axis ($z' = z$):

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



rotation about x':

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$



rotate about z again:

$$B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex

$$\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot\phi + \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \hat\theta + \begin{bmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{bmatrix} \dot\psi \quad (x, y, z)$$

$$\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot\psi + \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} \dot\theta + \begin{bmatrix} \sin\psi\sin\theta \\ \cos\psi\sin\theta \\ \cos\theta \end{bmatrix} \dot\phi \quad (x', y', z')$$

$$T = \frac{1}{2}\left(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2 \right)$$

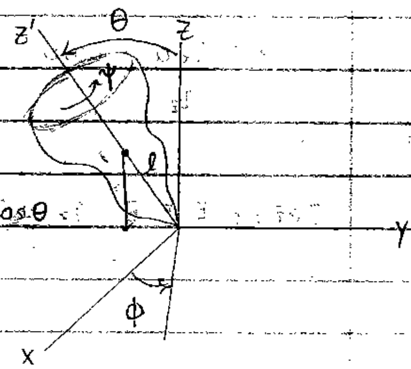
Write angular velocities (ω's) in the frame of the principle axes of the body so that products of inertia vanish

For body symmetric about z-axis (top), $I_{x'} = I_{y'}$

$$\therefore T = \frac{1}{2} I_{x'} (\omega_{x'}^2 + \omega_{y'}^2) + \frac{1}{2} I_{z'} \omega_{z'}^2$$

$$\omega_{x'}^2 + \omega_{y'}^2 = (\cos\psi\,\dot\theta + \sin\psi\sin\theta\,\dot\phi)^2 + (-\sin\psi\,\dot\theta + \cos\psi\sin\theta\,\dot\phi)^2$$
$$= \dot\theta^2 + \sin^2\theta\,\dot\phi^2$$

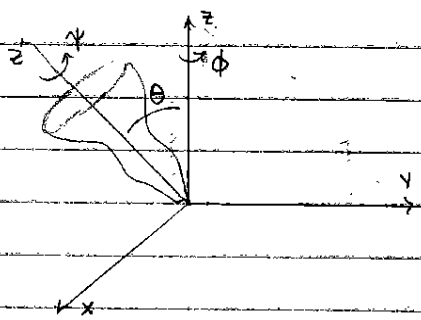
$$\omega_{z'}^2 = (\dot\psi + \dot\phi\cos\theta)^2$$



$V = Mgl\cos\theta$

10-31-05

$$L = \frac{I_{x'}}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{I_{z'}}{2}(\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta$$



Look for cyclic coordinates:

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_{z'}(\dot\psi + \dot\phi\cos\theta) = I_{z'}\omega_{z'} = I_{x'}a \quad (1)$$

defined a so that $I_{x'}a$ conserved

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = (I_{x'}\sin^2\theta + I_{z'}\cos^2\theta)\dot\phi + I_{z'}\cos\theta\,\dot\psi \quad (2)$$

$$= I_{x'}b$$ ← also defined b

a, b have units of angular momentum

Time doesn't appear explitity in L $\therefore \frac{dL}{dt} = 0$, Hamiltonian conserved

$$\therefore H = T + V = \frac{I_{x'}}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{I_{z'}}{2}(\dot\psi + \dot\phi\cos\theta)^2 + Mgl\cos\theta \quad (3)$$

$$I_{z'}\dot{\psi} = I_{x'}a - I_{z'}\dot{\phi}\cos\theta \qquad (4) \qquad \text{from (1)}$$

$$I_{x'}\dot{\phi}\sin^2\theta + I_{x'}a\cos\theta = I_{x'}b$$

$$\therefore \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

$$\dot{\psi} = \frac{I_{x'}a}{I_{z'}} - \cos\theta\left(\frac{b - a\cos\theta}{\sin^2\theta}\right)$$

Define $E' = E - \frac{I_{z'}\omega_{z'}^2}{2} = \frac{I_{x'}\dot{\theta}^2}{2} + \frac{I_{x'}\dot{\phi}^2\sin^2\theta}{2} + Mgl\cos\theta$

$$E' = \frac{I_{x'}\dot{\theta}^2}{2} + \frac{I_{x'}}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

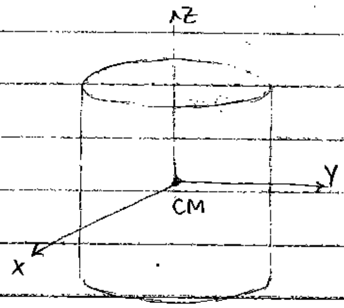
E' also conserved

$$E' = \frac{I_x\dot{\theta}^2}{2} + V'(\theta) \qquad V'(\theta) = Mgl\cos\theta + \frac{I_{x'}}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta}$$

As θ is varied, $V'(\theta)$ takes on various values and if $E' = V'(\theta)$ for some θ, $\dot{\theta} = 0$

Tutorial #7 11-1-05

Ex. What is the height-to-diameter ratio of a right cylinder such that the inertia ellipsoid at the center of the cylinder is a sphere?



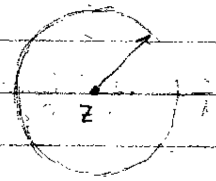
$I_z = I_x = I_y$ $\qquad dI_z = \frac{1}{2} m_{disk} r^2 = \frac{1}{2}\left(\frac{M\,dz}{h}\right) r^2 = \frac{1}{2}\frac{Mr^2}{h}dz$

$$\therefore I_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}\frac{Mr^2}{h}dz = \frac{1}{2}\frac{Mr^2}{h}\left(\frac{h}{2}\right) - \frac{1}{2}\frac{Mr^2}{h}\left(\frac{-h}{2}\right) = \frac{1}{2}Mr^2$$

$$dI_x = \frac{1}{4} m_{disk} r^2 + m_{disk} z^2 = \frac{1}{4}\left(\frac{M\,dz}{h}\right) r^2 + \left(\frac{M\,dz}{h}\right) z^2 = \frac{1}{4}\frac{Mr^2}{h}dz + \frac{Mz^2\,dz}{h}$$

$$\therefore I_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{1}{4}\frac{Mr^2}{h} + \frac{Mz^2}{h}\right)dz = \frac{1}{4}\frac{Mr^2}{h}\left(\frac{h}{2}+\frac{h}{2}\right) + \frac{1}{3}\frac{M}{h}\left(\frac{h^3}{8}+\frac{h^3}{8}\right) = \frac{1}{4}Mr^2 + \frac{1}{12}Mh^2$$

For a disk: $\quad I_z = \int \rho\, r\,dr\,d\theta\,(r^2) = \int \rho r^3\,dr\,d\theta = 2\pi\rho\frac{r^4}{4} = \frac{\pi\rho r^4}{2} = \frac{1}{2}r^2(\pi r^2\rho)$



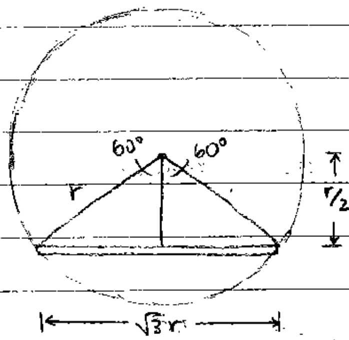
$$M = \int \rho\, r\,dr\,d\theta = 2\pi\rho\frac{r^2}{2} = \pi r^2\rho$$

$$\therefore I_z = \frac{1}{2}Mr^2$$

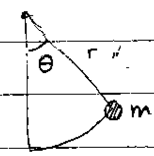
$$I_z = I_x \Rightarrow \frac{1}{2}Mr^2 = \frac{1}{4}Mr^2 + \frac{1}{12}Mh^2 \Rightarrow \frac{1}{4}Mr^2 = \frac{1}{12}Mh^2 \Rightarrow 3r^2 = h^2 \Rightarrow h = \sqrt{3}\,r$$

Goldstein 14.

①



②



A uniform rod slides with its ends on a smooth vertical circle. If the rod subtends an angle of 120° at the center of the circle, show that the equivalent simple pendulum has a length = r.

Same Lagrangian! $V = -Mg\frac{r}{2}\cos\theta$

$$T = \frac{1}{2} I \dot{\theta}^2$$

$\longleftarrow \ell \longrightarrow$ $\quad dI = m_{bit} r^2 = \left(\frac{M}{\ell}dx\right)x^2 = \frac{Mx^2}{\ell}dx$

$$I = \int_{-\ell/2}^{\ell/2} \frac{Mx^2}{\ell}dx = \frac{1}{3}\frac{Mx^3}{\ell}\Big|_{-\ell/2}^{\ell/2} = \frac{1}{3}\frac{M}{\ell}\left(\frac{\ell^3}{8}+\frac{\ell^3}{8}\right) = \frac{1}{12}M\ell^2$$

For $\ell = \sqrt{3}r$: $I = \frac{1}{12}M(\sqrt{3}r)^2 = \frac{1}{4}Mr^2$

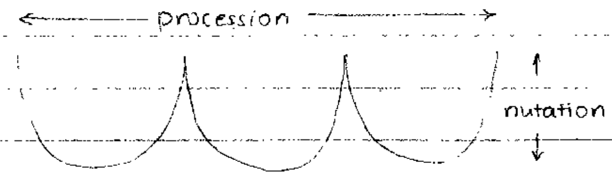
$\therefore I_{cm} = \left(\frac{1}{4}Mr^2\right) + M\left(\frac{r}{2}\right)^2 = \frac{1}{4}Mr^2 + \frac{Mr^2}{4} = \frac{1}{2}Mr^2$

$\therefore L_1 = \frac{1}{2}I\dot{\theta}^2 + Mg\frac{r}{2}\cos\theta = \frac{1}{4}Mr^2\dot{\theta}^2 + \frac{1}{2}Mgr\cos\theta$

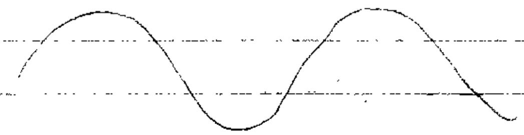
For pendulum: $L_2 = \frac{1}{2}mr\dot{\theta}^2 + mgr\cos\theta$

$L_1 = L_2$ for $M = 2m$!

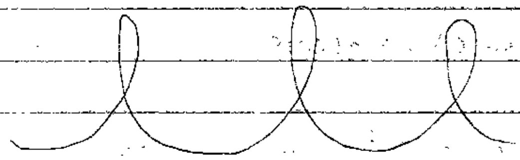
11-02-05

a) 

(Started at rest at a peak) → stops processing when at peak.

b) 

(started with velocity in same direction as top's motion)

c) 

(started against motion of top)

$$T = \frac{I_{x'}}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{I_{z'}}{2}(\dot\psi + \dot\phi\cos\theta)^2$$

$$V = Mgl\cos\theta$$

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_{z'}(\dot\psi + \dot\phi\cos\theta) = I_{z'}\omega_{z'} = I_{x'}a \qquad (1)$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = I_{x'}\dot\phi\sin^2\theta + I_{z'}\dot\phi\cos^2\theta + I_{z'}\cos\theta\dot\psi = (I_{x'}\sin^2\theta + I_{z'}\cos^2\theta)\dot\phi + I_{z'}\cos\theta\dot\psi$$
$$= I_{x'}b \qquad (2)$$

a, b are conserved quantities that were defined

$$I_z\dot\psi = I_{x'}a - I_{z'}\dot\phi\cos\theta \qquad \text{from (1)} \qquad (3)$$

$$I_{x'}\dot\phi\sin^2\theta + I_{x'}a\cos\theta = I_{x'}b \Rightarrow \dot\phi = \frac{b - a\cos\theta}{\sin^2\theta} \qquad (4)$$

Note: $\dot\phi = 0$ when $b - a\cos\theta = 0 \rightarrow \boxed{\cos\theta = \frac{b}{a} = u_0}$

Standard initial condition

how the top is usually started
(a) → at rest, $\dot\phi = 0$

$$\dot\psi = \frac{I_{x'} a}{I_{z'}} - \frac{I_{z'} \dot\phi \cos\theta}{I_{z'}} \qquad \text{from (3)}$$

$$= \frac{I_{x'} a}{I_{z'}} - \cos\theta \left(\frac{b - a\cos\theta}{\sin^2\theta} \right) \qquad \text{from(4)} \qquad (5)$$

$$E = T + V = \frac{I_{x'}}{2} (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{I_{z'} \omega_{z'}^2}{2} + Mgl\cos\theta$$

$$E' = E - \frac{I_{z'} \omega_{z'}^2}{2} = \frac{I_{x'}}{2} (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + Mgl\cos\theta$$

$$E' = \frac{I_{x'}}{2} \left(\dot\theta^2 + \left(\frac{b - a\cos\theta}{\sin^2\theta} \right)^2 \sin^2\theta \right) + Mgl\cos\theta \qquad \text{from (4)}$$

Note: E is conserved because $E = H$ and $\frac{\partial L}{\partial t} = 0$

$\frac{I_{z'} \omega_{z'}^2}{2}$ is conserved because $\frac{\partial L}{\partial \psi} = 0$

$\therefore$ E' conserved → a constant of the motion

Let $u = \cos\theta$ $\therefore$ $\dot u = -\sin\theta\, \dot\theta \Rightarrow \dot\theta^2 = \frac{\dot u^2}{\sin^2\theta} = \frac{\dot u^2}{1 - u^2}$

$$\therefore E' = \frac{I_{x'}}{2} \left(\frac{\dot u^2}{1 - u^2} + \frac{(b - au)^2}{1 - u^2} \right) + Mglu$$

(u : $+1 \to -1$ b/c $u = \cos\theta$)
$u = \cos\theta$ as θ goes from 0 to π

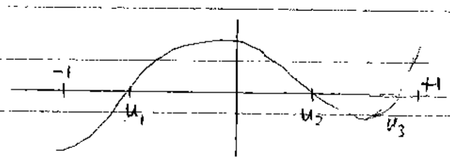
$$\Downarrow$$

$$(E' - Mglu)\left(\frac{2}{I_{x'}} \right) = \frac{\dot u^2 + (b - au)^2}{1 - u^2}$$

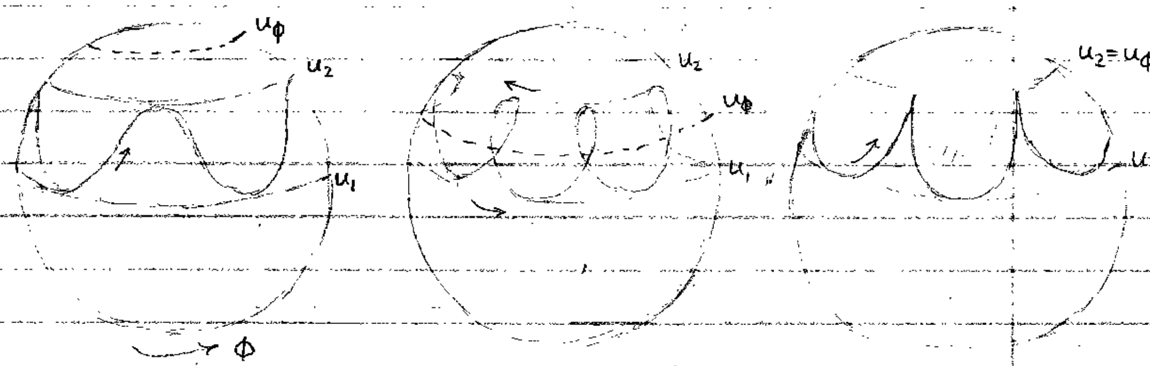
$$\Downarrow$$

$$\therefore \dot u^2 = (E' - Mglu)\left(\frac{2}{I_{x'}} \right)(1 - u^2) - (b - au)^2$$

highest power on RHS = 3
$\therefore$ 3 roots



$u_\phi = \frac{b}{a}$ can be any value between -1 and +1



$$\dot{u}^2 = (E' - Mg\ell u)\frac{2}{I_{x'}}(1-u^2) - (b-au)^2$$

If chose E' and $Mglu$ so that 1st term is very small, $\dot{u}$ would be imaginary!
(no roots for $\dot{u}$ between -1 and $+1$ $(-1 \le u \le +1)$)
or if at poles, $u = \pm 1$ and 1st term $= 0$ $\therefore$ $\dot{u}^2 < 0$ (imag. $\dot{u}$)

1st term maximized at $u = 0$ ($\cos\theta = 0 \to \theta = \pi/2$)

11-4-05

$$\dot{u}^2 = (E' - Mg\ell u)\frac{2}{I_{x'}}(1-u^2) - (b-au)^2$$

$\dot{\phi} = \dot{\theta} = 0$ at $t = t_0$: $\dot{\phi} = \dfrac{b - a\cos\theta}{\sin^2\theta} = 0 \Rightarrow \cos\theta_0 = \dfrac{b}{a} \Rightarrow \boxed{u_0 = \dfrac{b}{a}}$

(note $\dot{u} = 0$ when $\dot{\theta} = 0$ since $u = \cos\theta$)

$\dot{u} = -\sin\theta\,\dot{\theta}$ — One of the 3 roots of $\dot{u}$

$$\therefore \text{ at } t = t_0: \ \dot{u}^2 = \left(E' - Mg\ell\frac{b}{a}\right)\frac{2}{I_{x'}}\left(1 - \frac{b^2}{a^2}\right) - \left(b - a\left(\frac{b}{a}\right)\right)^2 = 0$$

$$\left(E' - Mg\ell\frac{b}{a}\right)\frac{2}{I_{x'}}\left(1 - \frac{b^2}{a^2}\right) = 0 \qquad \boxed{\therefore\ E' = Mg\ell\frac{b}{a}}$$

$\dot{\theta} = 0 \Rightarrow$ tip of the top not falling
$\dot{\phi} = 0 \Rightarrow$ not moving "sideways" E', a, b = constants of a motion

At all time: $\dot{u}^2 = \left(\frac{b}{a} - u\right) Mgl \frac{2}{I_{x'}} (1-u^2) - a^2\left(\frac{b}{a}-u\right)^2$

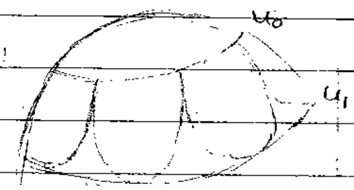
$u_0 = \frac{b}{a}$ $\qquad \dot{u}^2 = (u_0 - u) Mgl \frac{2}{I_{x'}}(1-u^2) - a^2(u_0-u)^2$

$$\therefore \dot{u}^2 = \underbrace{a^2(u_0-u)}_{\substack{\text{first factor} \\ \text{(first root} = u_0)}} \underbrace{\left[\frac{2Mgl}{I_{x'}a^2}(1-u^2) - (u_0-u)\right]}_{\text{value of } u \text{ that solves this expression} = 0 \text{ is another root}}$$

Take the limit of a fast top:

$$I_{x'}a^2 = \frac{I_{z'}}{I_{x'}} I_{z'}\omega_{z'}^2 \gg 2Mgl \quad \Rightarrow \quad T \gg V$$

$$\therefore \frac{2Mgl}{I_{x'}a^2}(1-u^2) \ll (u_0-u)$$



$u_0 - u_1 = \frac{2Mgl}{I_{x'}a^2}(1-u_1^2) \ll 1$ $\qquad$ Fast top: $u_0 \approx u_1$

to solve 2nd factor

$$\dot{u}^2 = a^2(u_0-u)[(u_0-u_1)-(u_0-u)] = a^2(u_0-u)(u_0-u_1-u_0+u)$$

* Since $u_0 \approx u_1$, can interchangeably have u_1 replace u_0
  ↳ the top doesn't nutate very much
  ∴ $(1-u^2) \sim$ constant (small range)

$$\dot{u}^2 = a^2(u_0-u)(u-u_1)$$

$$2\dot{u}\ddot{u} = a^2((u_0-u)\dot{u} - (u-u_1)\dot{u})$$

$$= a^2(u_0 - 2u + u_1)\dot{u} \quad \Rightarrow \quad \ddot{u} = a^2\left(\frac{u_0+u_1}{2} - u\right)$$

Define $y = u - \frac{u_0 + u_1}{2} \Rightarrow \ddot{y} = -a^2 y = \left(\frac{u_1 - u_0}{2}\right) \cos at$

$a = \frac{I_{z'} \omega_{z'}}{I_{x'}}$ $\quad \dot{\phi} = \frac{a(u_0 - u)}{\sin^2\theta}$, $\quad \phi = \frac{Mgl}{I_{z'}\omega_{z'}}(1 - \cos at)$

Harmonic Oscillator

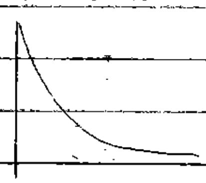
11-7-05

ODE: $m\ddot{x} + a\dot{x} + kx = 0 \Rightarrow$ substitute $x = e^{\alpha t}$

$\therefore \alpha^2 m x + \alpha a x + kx = 0 \longrightarrow m\alpha^2 + a\alpha + k = 0$

$$\alpha = \frac{-a \pm \sqrt{a^2 - 4mk}}{2m}$$

- if $a^2 > 4mk$



- if $a^2 < 4mk$: $\alpha = \frac{-a}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{a}{2m}\right)^2}$

$$\therefore x = \exp\left(\frac{-a}{2m}t\right)\left[A\cos\left(\sqrt{\frac{k}{m} - \left(\frac{a}{2m}\right)^2}\, t\right) + B\sin\left(\sqrt{\frac{k}{m} - \left(\frac{a}{2m}\right)^2}\, t\right)\right]$$

- Forced Oscillations: $m\ddot{x} + a\dot{x} + kx = Be^{i\omega t}$

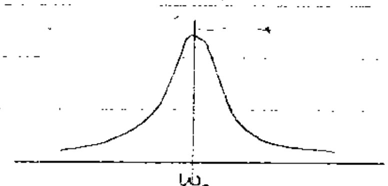
↳ substitute $x = Ae^{i\omega t}$ (since oscillation follows frequency of forcing, ω)

$-\omega^2 m A e^{i\omega t} + i\omega a A e^{i\omega t} + kAe^{i\omega t} = Be^{i\omega t}$

$\therefore -\omega^2 m A + i\omega a A + kA = B \rightarrow A(-\omega^2 m + i\omega a + k) = B$

$\therefore A = \frac{B}{k - \omega^2 m + i\omega a}$ $\quad$ let $\omega_0^2 = \frac{k}{m}$ $\rightarrow A = \frac{B}{m(\underbrace{\omega_0^2 - \omega^2}_{\text{real part}} + \underbrace{i\omega \frac{a}{m}}_{\text{imag part}})} = |A|e^{i\delta}$

$\therefore |A| = \frac{B}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{a\omega}{m}\right)^2}}$ $\quad \tan\delta = \frac{\frac{a\omega}{m}}{\omega_0^2 - \omega^2}$



maximum amplitude at $\omega \approx \omega_0$ (max $|A|$)

(Lorenz)

- nearly all sufficiently small oscillations are harmonic

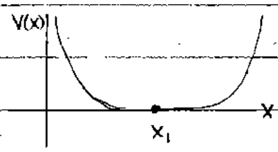
$L = \frac{1}{2} m\dot{x}^2 - V(x)$ Suppose that $x = x_1$ is an equilibrium $\left.\frac{\partial V}{\partial x}\right|_{x=x_1} = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + \frac{\partial V}{\partial x} = 0$ if at equilibrium at $x = x_1$, no force $\rightarrow m\ddot{x} = 0$

$$V = V(x_1) + (x - x_1)\left.\frac{\partial V}{\partial x}\right|_{x=x_1} + \frac{1}{2}(x-x_1)^2 \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_1} + \cdots$$

Let $q = x - x_1$: $L = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} q^2 \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_1}$ ($V(x_1)$ = constant, discard; $\partial V/\partial x|_{x=x_1} = 0$)

This fails if $\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_1} = 0$ eg. $V = \lambda(x-x_1)^4$



↳ For harmonic oscillator, frequency doesn't depend on amplitude

↳ in this case, it does ∴ fails

$L = \frac{1}{2} T_{ij}\dot{q}_i\dot{q}_j - V(q_1, \ldots, q_n)$ T_{ij} = matrix usually diagonal, not always

Let q_{0i} be an equilibrium position : $q_i = \underbrace{q_{0i}}_{\text{const}} + \eta_i$ ∴ $\dot{q}_i = \dot{\eta}_i$

$$V(q_1, \ldots, q_n) = V(q_{01}, \ldots, q_{0n}) + \sum_i \eta_i \left.\frac{\partial V}{\partial q_i}\right|_{q_0} + \sum_i \frac{\eta_i\eta_j}{2} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q_0}$$

$$\therefore V = \frac{\eta_i\eta_j}{2}\left.\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)\right|_{q_0} = \frac{1}{2} V_{ij}\eta_i\eta_j$$

$\therefore L = \frac{1}{2}(T_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j)$

$T_{ij}^{o} = T_{ij}(q_1^0, q_2^0, \ldots, q_n^0)$ → evaluated at equilibrium point

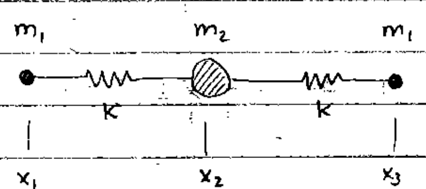
Applying Lagrange's eqn: $\sum_j \left(T_{ij}\ddot{\eta}_j + V_{ij}\eta_j\right) = 0$ $\boxed{\text{EOM}}$ one for each coord: i's

Try $\eta_j = C a_j e^{-i\omega t}$ $\Rightarrow$ $-\omega^2 T_{ij} a_j + V_{ij} a_j = 0$

$$A\vec{a} = 0 \qquad \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} \qquad A = \begin{bmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \cdots \\ \vdots & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

$\mathbb{V}\vec{a} = \omega^2 \mathbb{T}\vec{a} \Rightarrow [\mathbb{T}^{-1}\mathbb{V}]\vec{a} = \omega^2\vec{a}$ EIGENVALUE EQUATION

Tutorial 11-8-05



$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{x}_3^2 + \frac{1}{2}m_2\dot{x}_2^2$

$V = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2$

(CO_2 molecule)

Define $\eta_1^2 = m_1 x_1^2$ $\therefore T = \frac{1}{2}\dot{\eta}_1^2 + \frac{1}{2}\dot{\eta}_2^2 + \frac{1}{2}\dot{\eta}_3^2$ (now $\mathbb{T}$ is the identity matrix)

$\underbrace{\begin{aligned} \eta_2^2 &= m_2 x_3^2 \\ \eta_3^2 &= m_1 x_2^2 \end{aligned}}_{\text{mass-weighted coordinates}}$ $V = \frac{1}{2}k\left(\frac{\eta_1}{\sqrt{m_1}} - \frac{\eta_2}{\sqrt{m_2}}\right)^2 + \frac{1}{2}k\left(\frac{\eta_2}{\sqrt{m_2}} - \frac{\eta_3}{\sqrt{m_1}}\right)^2$

$$V = \frac{1}{2}k\left(\frac{\eta_1^2}{m_1} + \frac{\eta_2^2}{m_2} - \frac{2\eta_1\eta_2}{\sqrt{m_1 m_2}} + \frac{\eta_2^2}{m_2} + \frac{\eta_3^2}{m_1} - \frac{2\eta_2\eta_3}{\sqrt{m_1 m_2}}\right)$$

$$T = \sum_{i,j}\frac{1}{2}T_{ij}\dot{\eta}_i\dot{\eta}_j \qquad T_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$V_{ij} = \begin{bmatrix} \frac{k}{m_1} & -\frac{k}{\sqrt{m_1 m_2}} & 0 \\ -\frac{k}{\sqrt{m_1 m_2}} & \frac{2k}{m_2} & -\frac{k}{\sqrt{m_1 m_2}} \\ 0 & -\frac{k}{\sqrt{m_1 m_2}} & \frac{k}{m_1} \end{bmatrix}$$

$$V = \sum_{ij} \frac{1}{2} V_{ij} \eta_i \eta_j$$
$$= \frac{1}{2}\left[V_{11}\eta_1^2 + V_{12}\eta_1\eta_2 + V_{13}\eta_1\eta_3 + V_{21}\eta_2\eta_1 + V_{22}\eta_2^2 + V_{23}\eta_2\eta_3 + V_{31}\eta_3\eta_1 + V_{32}\eta_3\eta_2 + V_{33}\eta_3^2 \right]$$
$$= \frac{1}{2}\begin{bmatrix}\eta_1 & \eta_2 & \eta_3\end{bmatrix}\begin{bmatrix} V_{ij} \end{bmatrix}\begin{bmatrix}\eta_1 \\ \eta_2 \\ \eta_3\end{bmatrix}$$

V_{ij} must be symmetric because it comes from partial derivatives, and $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$

$$V_{11} = \frac{\partial V}{\partial \eta_1 \partial \eta_1} = \frac{\partial}{\partial \eta_1}\left(\frac{k\eta_1}{m_1}\right) = \frac{k}{m_1}$$

$$L = \frac{1}{2} T_{ij}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2} V_{ij}\eta_i\eta_j = \frac{1}{2}\mathbb{I}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2}V_{ij}\eta_i\eta_j$$

$$\frac{\partial L}{\partial \dot{\eta}_j} = \frac{1}{2}\mathbb{I}\dot{\eta}_i \qquad \frac{\partial L}{\partial \eta_i} = -\frac{1}{2}V_{ij}\eta_i \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_j}\right) - \frac{\partial L}{\partial \eta_j} = 0$$

$$\therefore \mathbb{I}\ddot{\eta}_i + V_{ij}\eta_i = 0$$

$$\eta_i = C a_i e^{-i\omega t} \Rightarrow -\omega^2 \mathbb{I} a_i + V_{ij} a_i = 0 \Rightarrow (\mathbb{V} - \omega^2 \mathbb{I})\vec{a} = 0$$

need to diagonalize V: ($\vec{a}$ is eigenvector, ω^2 is eigenvalue)

$$\det(V - \omega^2 \mathbb{I}) = \begin{vmatrix} \frac{k}{m_1} - \omega^2 & -\frac{k}{\sqrt{m_1 m_2}} & 0 \\ -\frac{k}{\sqrt{m_1 m_2}} & \frac{2k}{m_2} - \omega^2 & -\frac{k}{\sqrt{m_1 m_2}} \\ 0 & -\frac{k}{\sqrt{m_1 m_2}} & \frac{k}{m_1} - \omega^2 \end{vmatrix} = 0$$

$$= \left[\left(\frac{k}{m_1} - \omega^2\right)\left(\frac{2k}{m_2} - \omega^2\right)\left(\frac{k}{m_1} - \omega^2\right) - \left(\frac{k}{m_1} - \omega^2\right)\left(\frac{k^2}{m_1 m_2}\right) - \left(\frac{k}{m_1} - \omega^2\right)\left(\frac{k^2}{m_1 m_2}\right)\right]$$

$$\det(V - \omega^2 \mathbb{I}) = \left(\frac{k}{m_1} - \omega^2\right)\left[\left(\frac{2k}{m_2} - \omega^2\right)\left(\frac{k}{m_1} - \omega^2\right) - 2\left(\frac{k^2}{m_1 m_2}\right)\right]$$

$$= \left(\frac{k}{m_1} - \omega^2\right)\left[\frac{2k^2}{m_1 m_2} - \frac{2k\omega^2}{m_2} - \frac{k\omega^2}{m_1} + \omega^4 - \frac{2k^2}{m_1 m_2}\right]$$

$$= \left(\frac{k}{m_1} - \omega^2\right)\left[-\frac{2k\omega^2}{m_2} - \frac{k\omega^2}{m_1} + \omega^4\right]$$

$$\det(\mathbb{V}-\omega^2 I) = \left(\frac{k}{m_1}-\omega^2\right)\omega^2\left[\omega^2 - \frac{2k}{m_2} - \frac{k}{m_1}\right]$$

- I have a solution ($q_i^0(t)$) to the EOM $\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\Big|_{q_0} - \frac{\partial L}{\partial q_i}\Big|_{q_0} = 0$ 11-9-05

- Let's look at $q_i(t) = q_i^0(t) + \eta_i(t)$ (assume $\eta_i(t)$ small perturbation)

$$L = \underbrace{\left[L(q_i^0, \dot q_i^0) + \frac{\partial L}{\partial q_i}\Big|_{q_0}\eta_i + \frac{\partial L}{\partial \dot q_i}\Big|_{q_0}\dot\eta_i\right]}_{\text{"thrown out"}} + \frac{1}{2}\frac{\partial^2 L}{\partial q_i \partial q_j}\Big|_{q_0}\eta_i\eta_j + \frac{1}{2}\frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\Big|_{q_0}\dot\eta_i\dot\eta_j + \frac{\partial^2 L}{\partial q_i \partial \dot q_i}\Big|_{q_0}\eta_i\dot\eta_j + \dots$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\eta_i} - \frac{\partial L}{\partial \eta_i} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\Big|_{q_0} + \frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\Big|_{q_0}\dot\eta_j + \frac{\partial^2 L}{\partial q_j \partial \dot q_i}\Big|_{q_0}\eta_j\right) - \left(\frac{\partial L}{\partial q_i}\Big|_{q_0} + \frac{\partial^2 L}{\partial q_i \partial q_j}\Big|_{q_0}\eta_j + \frac{\partial^2 L}{\partial q_i \partial \dot q_j}\dot\eta_j\right) = 0$$

note: $\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\Big|_{q_0} - \frac{\partial L}{\partial q_i}\Big|_{q_0} = 0$ (Lagrangian of solution of EOM, $q_i^0(t)$)

Ex. $L = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2) + kr^\alpha$ potential (kr^α) is a central force

(eg gravity → $\alpha = -1$) ↑ pushing or pulling ~~from r=0~~

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = 0 \rightarrow \ddot r - r\dot\theta^2 - \alpha k r^{\alpha-1} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0 \rightarrow \frac{d}{dt}(r^2\dot\theta) = 0$$

choose solution to perturb about → take $\ddot r = 0$

$$\therefore r_0\dot\theta^2 = -\alpha k r_0^{\alpha-1} \Rightarrow \dot\theta^2 = -\alpha k r_0^{\alpha-2} = \Omega^2$$

$$\boxed{\begin{aligned}\therefore r &= r_0 + \eta_1(t)\\ \theta &= \Omega t + \eta_2(t)\end{aligned}}$$

$$L = \frac{1}{2}\left(\dot\eta_1^2 + (r_0+\eta_1)^2(\Omega+\dot\eta_2)^2\right) + k r_0^\alpha\left(1+\frac{\eta_1}{r_0}\right)^\alpha$$

$$L = \frac{1}{2}\left(\dot\eta_1^2 + (r_0^2 + 2r_0\eta_1 + \eta_1^2)(\Omega^2 + 2\Omega\dot\eta_2 + \dot\eta_2^2)\right) + k r_0^\alpha\left(1 + \alpha\frac{\eta_1}{r_0} + \frac{\alpha(\alpha-1)}{2}\left(\frac{\eta_1}{r_0}\right)^2\right)$$

- Drop 1st order terms since they sum to 0 due to being a solution of Lagrange's eqn

$$\therefore L = \frac{1}{2}\left(\dot\eta_1^2 + r_0^2\dot\eta_2^2 + 4r_0\Omega\eta_1\dot\eta_2 + \Omega^2\eta_1^2\right) + \underbrace{k r_0^{(\alpha-2)}}_{-\Omega^2}\frac{\alpha(\alpha-1)}{2}\eta_1^2$$

$$= \frac{1}{2}\left(\dot\eta_1^2 + r_0^2\dot\eta_2^2 + 4r_0\Omega\eta_1\dot\eta_2 + \Omega^2(\alpha-2)\eta_1^2\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\eta_1} - \frac{\partial L}{\partial\eta_1} = 0 \Rightarrow \overbrace{\frac{d}{dt}(\dot\eta_1)}^{\ddot\eta_1} - \left(2r_0\Omega\dot\eta_2 + \Omega^2(\alpha-2)\eta_1\right) = 0 \qquad (1)$$

$$\frac{d}{dt}\big(\underbrace{r_0^2\dot\eta_2 + 2r_0\Omega\eta_1}_{p_2\ (\text{define})}\big) = 0 \qquad \dot\eta_2 = \frac{p_2 - 2r_0\Omega\eta_1}{r_0^2} \qquad (2)$$

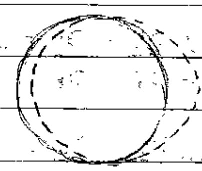
- Sub (2) into (1): $\ddot\eta_1 = 2r_0\Omega\left(\frac{p_2 - 2r_0\Omega\eta_1}{r_0^2}\right) - (\alpha-2)\Omega^2\eta_1$

$$= \frac{2\Omega p_2}{r_0} - 4\Omega^2\eta_1 - (\alpha-2)\Omega^2\eta_1$$

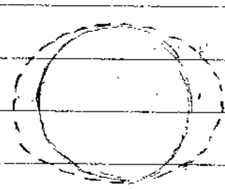
$$\boxed{\ddot\eta_1 = \frac{2\Omega p_2}{r_0} - (\alpha+2)\Omega^2\eta_1} = -(\alpha+2)\Omega^2\left[K + \eta_1\right]$$

$\eta_1 = A\cos(\sqrt{\alpha+2}\,\Omega t) + K$

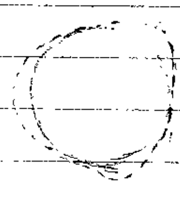
$\alpha = -1 \rightarrow \sqrt{\alpha+2} = 1$ $\therefore$ frequency of perturbed motion is same as original motion



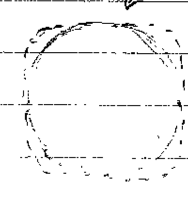
$\alpha = 2 \rightarrow \sqrt{\alpha+2} = 2$



$\alpha = 7 \rightarrow \sqrt{\alpha+2} = 3$



$\alpha = 14$



Pendulum Example (1-dimensional)

11-14-05

$L = L_0 + \alpha L_I$ — We know that $q_0(t)$ solves L_0

— Find an approximate solution for L: $q(t) = q_0(t) + \alpha\eta(t)$

(α is small)

$$L = L_0\left(q_0(t) + \alpha\eta(t);\ \dot{q}_0(t) + \alpha\dot{\eta}(t)\right) + \alpha L_I\left(q_0 + \alpha\eta;\ \dot{q}_0 + \alpha\dot{\eta}\right)$$

$$L = L_0(q_0, \dot{q}_0) + \alpha \frac{\partial L_0}{\partial q}\bigg|_{q_0}\eta + \alpha\frac{\partial L_0}{\partial \dot{q}}\bigg|_{q_0}\dot{\eta} + \frac{\alpha^2}{2}\left[\frac{\partial^2 L_0}{\partial q^2}\bigg|_{q_0}\eta^2 + 2\frac{\partial^2 L_0}{\partial \dot{q}\partial q}\bigg|_{q_0}\eta\dot{\eta} + \frac{\partial^2 L_0}{\partial \dot{q}^2}\bigg|_{q_0}\dot{\eta}^2\right]$$
$$+ \alpha\left[L_I(q_0) + \alpha\frac{\partial L_I}{\partial q}\eta + \frac{\partial L_I}{\partial \dot{q}}\dot{\eta}\right]$$

(letting $\alpha = 1$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) - \frac{\partial L}{\partial \eta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{q}}\bigg|_{q_0} + \frac{\partial^2 L_0}{\partial \dot{q}\partial q}\bigg|_{q_0}\eta + \frac{\partial^2 L_0}{\partial \dot{q}^2}\bigg|_{q_0}\dot{\eta} + \frac{\partial L_I}{\partial \dot{q}}\right)$$
$$- \frac{\partial L_0}{\partial q}\bigg|_{q_0} - \frac{\partial^2 L_0}{\partial q^2}\bigg|_{q_0}\eta - \frac{\partial^2 L_0}{\partial \dot{q}\partial q}\bigg|_{q_0}\dot{\eta} - \frac{\partial L_I}{\partial q}\bigg|_{q_0} = 0$$

$$L' = \frac{1}{2}\left[\frac{\partial^2 L_0}{\partial q^2}\bigg|_{q_0}\eta^2 + 2\frac{\partial^2 L_0}{\partial \dot{q}\partial q}\bigg|_{q_0}\dot{\eta}\eta + \frac{\partial^2 L_0}{\partial \dot{q}^2}\bigg|_{q_0}\dot{\eta}^2\right] + \frac{\partial L}{\partial q}\bigg|_{q_0}\eta + \frac{\partial L}{\partial \dot{q}}\bigg|_{q_0}\dot{\eta}$$

— Pendulum: $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

$$L' = \frac{1}{2}\dot{\theta}^2 + \omega_0^2\cos\theta = \frac{1}{2}\dot{\theta}^2 + \omega_0^2\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - + \ldots\right)$$

$L_0 = \frac{1}{2}\dot{\theta}^2 - \frac{\omega_0^2\theta^2}{2}$ (dropped constant term from Lagrangian) $L_1 = \frac{\omega_0^2\theta^4}{24}$

$$\frac{d}{dt}(\dot{\theta}) + \omega_0^2\theta = 0 \rightarrow \ddot{\theta} = -\omega_0^2\theta \rightarrow \theta_0 = A\sin(\omega_0 t)$$

$$\frac{\partial^2 L_0}{\partial \theta^2} = -\omega_0^2 \quad \frac{\partial^2 L_0}{\partial \dot{\theta}^2} = 1 \quad \frac{\partial^2 L_0}{\partial\theta\partial\dot{\theta}} = 0 \quad \frac{\partial L_1}{\partial \theta} = \frac{\omega_0^2\theta^3}{6} \quad \frac{\partial L_1}{\partial \dot{\theta}} = 0$$

perturbed Lagrangian: $L' = \frac{1}{2}(\dot{\eta}^2 - \omega_0^2 \eta^2) + \frac{\omega_0^2 \theta_0^3 \eta}{6}$

$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\theta}}\right) - \frac{\partial L'}{\partial \theta} = 0 \Rightarrow \ddot{\eta} + \omega_0^2 \eta - \frac{\omega_0^2}{6}[\theta_0(t)]^3 = 0$ Driven harmonic oscillator

$[\theta_0(t)]^3 = A^3 \sin^3(\omega_0 t)$

$= A^3 \frac{1}{4}(3\sin(\omega_0 t) - \sin(3\omega_0 t))$

(*)

$\ddot{\eta} + \omega_0^2 \eta = \frac{A^3 \omega_0^2}{24}(3\sin(\omega_0 t) - \sin(3\omega_0 t))$ ⇒ if driven at ω_0 (natural frequency), perturbation grows; doesn't make sense

2nd term is OK: amplitude $= C = \frac{1}{192} A^3$ $\eta = C\sin(3\omega_0 t)$

Substitute $\theta = A\sin([\omega_0 + \omega_1]t)$ into Lagrange's eqn: (slightly "wrong" solution, ω_1 small)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 2\omega_1 \omega_0 A \sin(\omega t)$ "extra bit"

So subtract from (*):

$\ddot{\eta} + \omega_0^2 \eta = \frac{A^3 \omega_0^2}{24}(3\sin(\omega_0 t) - \sin(3\omega_0 t)) - 2\omega_1 \omega_0 A \sin(\omega t)$

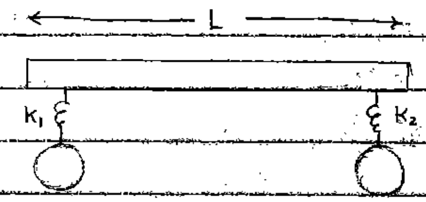
pick $\omega_1 = \frac{-A^2 \omega_0}{16}$

* the period of pendulum depends on amplitude

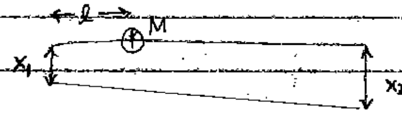
∴ change frequency of solution by a little bit to indicate the change in period (small ω_1)

Tutorial 11-15-05

L



Loaded:



$\Delta \ell_2 = x_2$

if symmetric: $\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}$, want $\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$

2. 3 degrees of freedom → hold m_2 constant, change r and θ

→ hold m_1 constant, change ℓ (up and down)

$\ell = \ell_0 + \eta_1$

$\theta = \omega t + \eta_2$

$r = r_0 + \eta_3$

Ex. Orbit of Mercury is most significantly not closed, so point where it is closes to the sun shifts over centuries (orbit is eccentric)

$$V = \frac{-GMm}{r - r_g} \qquad r_g = 3\,\text{km} = \frac{2GM_\odot}{c^2}$$

How far off does Mercury miss closing its orbit?

$$L = \frac{1}{2} m(\dot r^2 + r^2\dot\theta^2) + \frac{GMm}{r - r_g}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow m\ddot r - mr\dot\theta^2 + \frac{GMm}{(r-r_g)^2} = 0$$

Let $\omega = \dot\theta$ (constant), $r = r_0$ (constant for the steady motion) → $\ddot r = 0$

$$mr_0\omega^2 = \frac{GMm}{(r-r_g)^2} \Rightarrow \omega^2 = \frac{GM}{(r-r_g)^2 r_0}$$

Perturbation: $r = r_0 + \eta_1$

$\theta = \omega t + \eta_2$

$$\frac{-GMm}{r - r_g} = \frac{-GMm}{r_0 + \eta_1 - r_g} = \frac{-GMm}{r_0 - r_g}\left(1 + \frac{\eta_1}{r_0 - r_g}\right)^{-1} = \frac{-GMm}{r_0 - r_g}\left(1 - \frac{\eta_1}{r_0 - r_g} + \frac{\eta^2}{(r_0 - r_g)^2} + \dots\right) = V$$

$$L = \frac{1}{2} m \left((\dot{r}_0 + \dot{\eta}_1)^2 + (r_0+\eta_1)^2(\omega + \dot{\eta}_2)^2 + \frac{GMm}{r_0 - r_g} \left(\quad \right) \right)$$ note $\dot{r}_0 = 0$

$$L^{(2)} = \frac{1}{2} m \left(\dot{\eta}_1^2 + r_0^2 \dot{\eta}_2^2 + 2r_0\eta_1(2\omega\dot{\eta}_2) + \eta_1^2\omega^2 \right) + \frac{GMm}{r_0 - r_g} \frac{\eta_1^2}{(r_0-r_g)^2}$$ Only 2nd order terms

$$L^{(2)} = \frac{1}{2} \left(\dot{\eta}_1^2 + r_0^2 \dot{\eta}_2^2 + 2r_0\eta_1(2\omega\dot{\eta}_2) + \eta_1^2\omega^2 \right) + \omega^2 \frac{r_0 \eta_1^2}{r_0 - r_g}$$

$$\frac{d}{dt}\left(\frac{\partial L^{(2)}}{\partial \dot{\eta}_2} \right) - \left(\frac{\partial L^{(2)}}{\partial \eta_2} \right) = 0 \rightarrow \frac{d}{dt}\left(r_0^2 \dot{\eta}_2 + 2r_0\eta_1\omega \right) = 0$$

$p_2 = 2r_0\omega\eta_1 + r_0^2\dot{\eta}_2$ $\quad \eta$

(conserved)

PHYS350

Lagrangian of a Continuum 11-16-05

$$L\left(\phi, \frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}\right) = \int dx \underbrace{\mathcal{L}\left(\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}\right)}_{\text{Lagrangian density}}$$

Action: $S = \int dx\, dt\, \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial t}\right)$ Find ϕ to minimize action, S

Let $\phi = \phi_0 + \alpha\eta$ $\eta = \eta(x,t)$ → position and time (small perturbation)

ϕ_0 = solution that minimizes S

$$\delta S = \int dt \int dx \left[\mathcal{L}\left(\phi_0, \frac{\partial\phi_0}{\partial x}, \frac{\partial\phi_0}{\partial t}\right) + \alpha\frac{\partial\mathcal{L}}{\partial\phi}\eta + \alpha\frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\eta + \alpha\frac{\partial\mathcal{L}}{\partial\phi'}\eta' - \mathcal{L}\right]$$

$\dot\phi = \frac{\partial\phi}{\partial t}$, $\phi' = \frac{\partial\phi}{\partial x}$

$$= \alpha\int dt \int dx \left[\left(\frac{\partial\mathcal{L}}{\partial\phi}\eta - \left(\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\phi'}\right)\eta\right) + \frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\eta\right]$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\eta\right) = \frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\eta + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right)\eta = 0$$

$$\delta S = \alpha\int dt \int dx \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right) - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right)\right]\eta$$

$$\boxed{\therefore\ \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right) + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0}$$

$$\frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial y}\right)} + \frac{d}{dz}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial z}\right)} + \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

for 3-D: can see t is just like the coords x, y, z

Ex. Energy density in an electric field: $\frac{1}{8\pi}E^2 = \frac{1}{8\pi}(\nabla\phi)^2$ $E = -\nabla\phi$

$$\text{Energy} = \int dV\left(\frac{1}{8\pi}(\nabla^2\phi) + \rho\phi\right)$$

← Lagrangian for electrostatics

$$= \int dx\,dy\,dz\left(\frac{1}{8\pi}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + \rho\phi\right)$$

minimize energy

$$\frac{1}{8\pi}\left[2\frac{d}{dx}\left(\frac{\partial\phi}{\partial x}\right) + 2\frac{d}{dy}\left(\frac{\partial\phi}{\partial y}\right) + 2\frac{d}{dz}\left(\frac{\partial\phi}{\partial z}\right)\right] - \rho = 0$$

$$\therefore \frac{1}{4\pi}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)=\rho \quad \Rightarrow \quad \boxed{\nabla^2\phi = 4\pi\rho}$$

Gauss' Law for E & M (configuration minimizing energy)

Ex. Potential energy of a string stretched (under tension):

$$V=\frac{1}{2}\lambda\int\left(\frac{\partial y}{\partial x}\right)^2 dx \qquad T=\frac{1}{2}\int\left(\frac{\partial y}{\partial t}\right)^2\frac{M}{L}\,dx$$

Let $\mu = \frac{M}{L}$ $\quad\lambda$ = tension of string

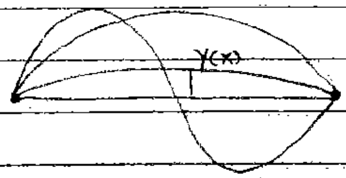
Action: $S=\int dt\,(T-V) = \frac{1}{2}\int dx\int dt\left(\mu\left(\frac{\partial y}{\partial t}\right)^2-\lambda\left(\frac{\partial y}{\partial x}\right)^2\right)$

$$\mu\frac{d}{dt}\left(\frac{\partial y}{\partial t}\right)-\lambda\frac{d}{dx}\left(\frac{\partial y}{\partial x}\right)=0 \quad\Rightarrow\quad \frac{\partial^2 y}{\partial t^2}-\frac{\lambda}{\mu}\frac{\partial^2 y}{\partial x^2}=0$$

WAVE EQ!

a particular solution: $y=\sin\left(\frac{n\pi}{L}x\right)\cos(\omega t)$ $\quad y(0)=y(L)=0$ as defined in problem

Subst.: $-\omega^2 y+\frac{\lambda}{\mu}\left(\frac{n\pi}{L}\right)^2 y=0 \rightarrow \omega^2=\frac{\lambda}{\mu}\left(\frac{n\pi}{L}\right)^2$



$$\frac{\partial^2 y}{\partial t^2} - \frac{\lambda}{\mu}\frac{\partial^2 y}{\partial x^2} = 0$$

$$y = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

$$-\omega_n^2 A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) + \frac{\lambda}{\mu}\left(\frac{n\pi}{L}\right)^2 A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t) = 0$$

$$\omega_n^2 = \frac{\lambda}{\mu}\left(\frac{n\pi}{L}\right)^2$$

$$T = \frac{1}{2}\int\left(\frac{\partial z}{\partial t}\right)^2 \frac{M}{A}\, dA \qquad \mu = \frac{M}{A}, \quad dA = dx\,dy \text{ for rect. coords.}$$

$$dA = r\,dr\,d\theta \text{ for polar coords.}$$

$$V = \frac{1}{2}\lambda \int\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] dA$$

$$L = \frac{1}{2}\left[\int dA \left(\mu\left(\frac{\partial z}{\partial t}\right)^2 - \lambda\left(\frac{\partial z}{\partial x}\right)^2 - \lambda\left(\frac{\partial z}{\partial y}\right)^2\right)\right] \qquad \mathcal{L}_{rect.} = \frac{1}{2}\mu \dot{z} - \frac{1}{2}\lambda(\nabla z)^2$$

$$\mathcal{L}_{polar} = r\left(\frac{1}{2}\mu \dot{z}^2 - \frac{1}{2}\lambda(\nabla z)^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial z}{\partial t}\right)}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial z}{\partial x}\right)}\right) + \frac{d}{dy}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial z}{\partial y}\right)}\right) - \frac{\partial \mathcal{L}}{\partial z} = 0$$

$$\mu \frac{d}{dt}\left(\frac{\partial z}{\partial t}\right) - \lambda\left(\frac{d}{dx}\left(\frac{\partial z}{\partial x}\right) + \frac{d}{dy}\left(\frac{\partial z}{\partial y}\right)\right) = 0$$

$$\mu\frac{\partial^2 z}{\partial t^2} - \lambda\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) = 0 \quad \Rightarrow \quad \mu\frac{\partial^2 u}{\partial t^2} - \lambda\nabla^2 z = 0$$

$$z = \sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{n\pi y}{L_y}\right)\cos(\omega_{n,m} t)$$

$$-\omega^2 + \frac{\lambda}{\mu}\left(\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2\right) = 0$$

Harmonics : $\sqrt{1+1} = \sqrt{2}$, $\sqrt{1+4} = \sqrt{5}$, $\sqrt{4+4} = \sqrt{8}$, $\sqrt{1+9} = \sqrt{10}$, $\sqrt{4+9} = \sqrt{13}$,

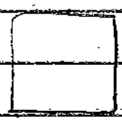
$1\sqrt{2}$ $\qquad$ $2\sqrt{2}$

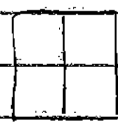
$\sqrt{1+16} = \sqrt{17}$, $\sqrt{9+9} = \sqrt{18}$, ...

$3\sqrt{2}$

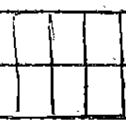
Nodal diagrams (nodes are where the point is not moving with time)

where the excitation = 0

n=1
m=1


n=1
m=2


n=2
m=2


m=3
n=1


(draw to scale)

$$y(x,t) = \sum A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right)$$

$$y(x,0) = y_0(x) = \sum A_n \sin\left(\frac{n\pi x}{L}\right)$$

ratio of coeff. indicate quality of sound (for violin, first A_n is larger than later ones , for sax, don't decrease as quickly

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) y_0(x)\,dx = \sum \int_0^L A_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\left.\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\end{aligned}\right\} \quad \sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$$

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) y_0(x)\,dx = \sum \int_0^L A_n\,dx\,\frac{1}{2}\left(\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right) = \frac{A_m L}{2}$$

Hamilton's Equations 11-21-05

$$L = L(q_i, \dot{q}_i, t) \Rightarrow dL = \sum_i \left[dq_i \frac{\partial L}{\partial q_i} + d\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right] + \frac{\partial L}{\partial t} dt$$

$\frac{\partial L}{\partial \dot{q}_i} = p_i$ = generalized momentum

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - F_i \qquad \text{Lagrange's eqn: } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

$$\therefore dL = \sum_i \left[dq_i(\dot{p}_i - F_i) + d\dot{q}_i p_i \right] + \frac{\partial L}{\partial t} dt$$

$$d\dot{q}_i p_i = ? \qquad d(p_i \dot{q}_i) = p_i d\dot{q}_i + \dot{q}_i dp_i \Rightarrow d\dot{q}_i p_i = d(p_i \dot{q}_i) - \dot{q}_i dp_i$$

$$\therefore dL = \sum_i \left[dq_i(\dot{p}_i - F_i) + d(p_i\dot{q}_i) - \dot{q}_i dp_i \right] + \frac{\partial L}{\partial t} dt$$

$$d(L - p_i\dot{q}_i) = \sum_i \left[dq_i(\dot{p}_i - F_i) - \dot{q}_i dp_i \right] + \frac{\partial L}{\partial t} dt$$

$$\therefore dH = \sum_i \left[dq_i(F_i - \dot{p}_i) + \dot{q}_i dp_i \right] - \frac{\partial L}{\partial t} dt \qquad (1)$$

– we can change H by changing q_i (coordinate), p_i (momentum), t (time)

$\therefore H = H(q_i, p_i, t)$

$$dH = \sum_i \left[\left(\frac{\partial H}{\partial q_i}\right) dq_i + \left(\frac{\partial H}{\partial p_i}\right) dp_i \right] + \left(\frac{\partial H}{\partial t}\right) dt \qquad \text{by defn. of partials}$$

$$= \sum_i \left[(F_i - \dot{p}_i) dq_i + \dot{q}_i dp_i \right] - \frac{\partial L}{\partial t} dt \qquad \text{from (1)}$$

$$\boxed{\therefore \frac{\partial H}{\partial q_i} = F_i - \dot{p}_i \qquad \frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}} \qquad \text{Hamilton's equations}$$

Hilroy

(1) Write out L.

(2) Calculate the momentum

(3) Calculate H

(4) Invert momentum to get $\dot{q}_i = \dot{q}_i(p_i)$

(5) Get $H = H(q_i, p_i, t)$

(6) Apply Hamilton's eqn.

Ex. (the pendulum) $L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$

$$H = T + V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta$$

$p_\theta = ml^2\dot{\theta} \Rightarrow H = p_\theta \dot{\theta} - L = ml^2\dot{\theta}^2 - \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta = \frac{1}{2} ml^2\dot{\theta}^2 - mgl\cos\theta$

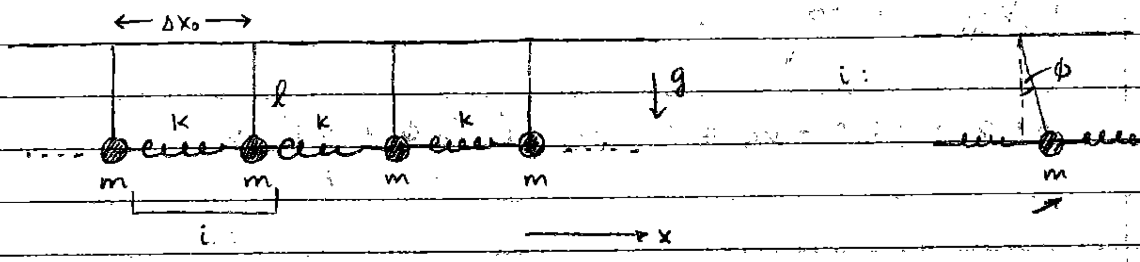
invert $p_\theta \rightarrow \dot{\theta} = \frac{p_\theta}{ml^2}$ $\quad \therefore H = \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl\cos\theta$

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$, $\quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \quad (F_\theta = 0)$

$\therefore \ddot{\theta} = \frac{\dot{p}_\theta}{ml^2} = \frac{-mgl\sin\theta}{ml^2} = -\frac{g}{l}\sin\theta$

Tutorial 11-22-05

Ex. An infinite chain of pendulums connected with springs.



$$T_i = \frac{1}{2} m\ell^2 \dot{\phi}_i^2 \qquad V_i = V_g + V_{spring}$$

$$V_g^i = mgh_i = mg\ell(1-\cos\phi) \qquad \cos\phi = \sqrt{1-\sin^2\phi}$$

$$= mg\ell(1-\sqrt{1-\phi^2}) \qquad \sin\phi \approx \phi \text{ for } \phi \text{ small}$$

$$\Delta x = \ell\sin\phi_i - \ell\sin\phi_{i-1} \qquad = mg\ell\left(1-\left(1-\frac{\phi^2}{2}\right)\right) \qquad \sqrt{1-\phi^2} \approx 1-\frac{\phi^2}{2}$$

$$= \ell\phi_i - \ell\phi_{i-1} \qquad = \frac{mg\ell\phi_i^2}{2}$$

$$V_{spring} = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}k\ell^2(\phi_i - \phi_{i-1})^2$$

$$L = \sum_i \frac{1}{2}\left(m\ell^2\dot{\phi}_i^2 - k\ell^2(\phi_i-\phi_{i-1})^2 - mg\ell\phi_i^2\right)$$

Define $\mu = \frac{m}{l_{spring}} = \frac{m}{dx} \Rightarrow \boxed{m = \mu dx}$ (μ = ratio of mass to dist. b/t masses)

$$\boxed{\phi_i - \phi_{i-1} = \frac{\partial\phi}{\partial x}dx} \qquad \phi_i = \phi(x_i) \qquad \boxed{\lambda = kdx}$$

$$\therefore L = \sum_i \frac{1}{2}\left(\ell^2\mu\dot{\phi}^2 dx - k\ell^2\left(\frac{\partial\phi}{\partial x}\right)^2(dx)^2 - \mu g\ell\phi^2 dx\right)$$

$$= \sum_i \frac{1}{2}\left(\ell^2\mu\dot{\phi}^2 dx + \lambda\ell^2\left(\frac{\partial\phi}{\partial x}\right)^2 dx - \mu g\ell\phi^2 dx\right)$$

$$\therefore L = \frac{1}{2}\int\left(\mu\ell^2\dot{\phi}^2 - \lambda\ell^2\phi'^2 - \mu g\ell\phi^2\right)dx \Rightarrow \mathcal{L} = \frac{1}{2}\left(\mu\ell^2\dot{\phi}^2 - \lambda\ell^2\phi'^2 - \mu g\ell\phi^2\right)$$

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$-\lambda l^2 \phi'' + \mu l^2 \ddot{\phi} + \mu g l \phi = 0 \Rightarrow \ddot{\phi} - \frac{\lambda}{\mu}\phi'' + \frac{g}{l}\phi = 0$$

- Taking the following limits:

  $g \to 0$: $\ddot{\phi} - \frac{\lambda}{\mu}\phi'' = 0$ WAVE EQN

  $\lambda \to 0$: $\ddot{\phi} + \frac{g}{l}\phi = 0$ PENDULUM EOM

- Klein-Gordon Equation (massive scalar field)

= Guess $\phi = \phi_0 \cos(kx - \omega t)$ $\quad \ddot{\phi} = -\omega^2 \phi_0 \cos(kx - \omega t)$

$\phi'' = -k^2 \phi_0 \cos(kx - \omega t)$

$$\therefore -\omega^2 \phi_0 \cos(kx-\omega t) + \frac{\lambda}{\mu} k^2 \phi_0 \cos(kx-\omega t) + \frac{g}{l}\phi_0 \cos(kx-\omega t) = 0$$

$\boxed{\therefore \omega^2 = \frac{\lambda}{\mu} k^2 + \frac{g}{l}}$ Dispersion relation

$\boxed{\omega^2 = \frac{\lambda}{\mu} k^2 + \omega_0^2}$ for $\omega_0 = \sqrt{\frac{g}{l}}$

(For light, $\omega = ck^2$)

For $\omega \gg \omega_0 \rightarrow \omega = \left(\frac{\lambda}{\mu}\right)\sqrt{\frac{2\pi}{\lambda}}$

$$k = \left[(\omega^2 - \omega_0^2)\frac{\mu}{\lambda}\right]^{1/2}$$

Poisson Brackets

11-23-05

$f(q,p,t) \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q}$

Hamilton's equations: $\dot{p} = -\frac{\partial H}{\partial q}$, $\dot{q} = \frac{\partial H}{\partial p}$

$\therefore \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} = \frac{\partial f}{\partial t} + [H,f]$ (*)

Defn: Poisson Bracket: $[H,f] = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}$ (called commutator in QM)

if $\frac{\partial f}{\partial t} = 0$, then $\frac{df}{dt} = [H,f]$

- Properties: ① $[f,g] = -[g,f]$
  ② $[f,c] = 0$
  ③ $[af_1+bf_2, g] = a[f_1,g] + b[f_2,g]$
  ④ $[f_1f_2, g] = f_1[f_2,g] + f_2[f_1,g]$

$[q,p] = \frac{\partial q}{\partial p}\frac{\partial p}{\partial q} - \frac{\partial q}{\partial q}\frac{\partial p}{\partial p} = -1$

$\left.\begin{array}{l}[q_i,q_j] = 0 \\ [p_i,p_j] = 0\end{array}\right\}$ i,j independent

$[p_i,q_j] = \delta_{ij}$ (=1 when i=j, =0 if i≠j)

$\frac{d}{dt}[f,g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] + \left[f, [H,g]\right] + \left[[H,f], g\right]$ from (*)

- If neither f nor g are explicit functions of t (if both are integrals of the motion), then $[f,g]$ is too.

- If 2 angular momenta are conserved, the 3rd must also be conserved (calculate Poisson bracket of first 2)

$[A,B] = AB - BA$, A and B commute if $[A,B] = 0$ defined this way in QM

$[\hat{p}_x, \hat{x}] = -i\hbar$

$[p,f] = \frac{\partial p}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial p}{\partial q}\frac{\partial f}{\partial p} = (1)\frac{\partial f}{\partial q} - (0)\frac{\partial f}{\partial p} = \frac{\partial f}{\partial q}$

So $[\hat{p}_x, f] = -i\hbar\frac{\partial}{\partial x} f$ $\quad -i\hbar\frac{\partial}{\partial x}$ is the momentum operator $\hat{p}_x$

$[\hat{p}_x, f]\Psi = \hat{p}_x(f\Psi) - f(\hat{p}_x\Psi) = -i\hbar\frac{\partial f}{\partial x}\Psi$ try $\boxed{\hat{p}_x = -i\hbar\frac{\partial}{\partial x}}$

$= -i\hbar\left(\frac{\partial}{\partial x}(f\Psi) - f\left(\frac{\partial\Psi}{\partial x}\right)\right)$

Assumption is that $\underbrace{[A,B]_Q}_{\text{operators in Quantum Mechanics}} = AB - BA = -i\hbar\underbrace{[A,B]}_{\text{functions in Classical mechanics}} = -i\hbar\left(\frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial p}\right)$

$[\hat{p}_x, f]_Q\Psi = -i\hbar\left(\frac{\partial f}{\partial x}\Psi + f\frac{\partial\Psi}{\partial x} - f\left(\frac{\partial\Psi}{\partial x}\right)\right) = -i\hbar\frac{\partial f}{\partial x}\Psi = -i\hbar[p,f]\Psi$

$[H,f]_c = \frac{df}{dt}$ $\left(\text{assuming } \frac{\partial f}{\partial t} = 0\right) \Rightarrow [H,f]_Q = -i\hbar\frac{\partial f}{\partial t}$ $\boxed{\hat{H} = i\hbar\frac{\partial}{\partial t}}$

$\frac{\partial}{\partial x}(fg) \neq f\frac{\partial g}{\partial x}$ derivative doesn't commute

$H = T + V = \frac{p^2}{2m} + V \Rightarrow \hat{H} = T + V = \frac{\hat{p}^2}{2m} + V$ mult. by Ψ

$\therefore i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi$ Schrodinger's eqn

Phase Space

11-25-05

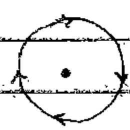
- Phase space is a function of $(q_1, q_2, q_3, \ldots, q_n, p_1, p_2, p_3, \ldots, p_n)$
- From Hamilton's equations: $\dot{q}_1 = \frac{\partial H}{\partial p_1}$ $\quad \dot{p}_1 = F_1 - \frac{\partial H}{\partial q_1}$ $\quad \frac{\partial H}{\partial t} = 0, \frac{\partial F}{\partial t} = 0, F(q,p)$

Ex. Phase space for a pendulum:

$$\dot{p}_\theta = -mg\ell \sin\theta \qquad \dot{\theta} = \frac{p_\theta}{m\ell^2}$$

→ plot momentum vs. position for each coordinate (eg. θ)
→ top line represents pushing so hard that the pendulum swings completely over (higher momentum at bottom of swing, lowest momentum at top, never have $p=0$)
→ outmost closed curve represents a pendulum that almost swings all the way around ($p=0$ when $\theta \sim 3.14$ → at $\theta = \pi$, pendulum at top, $p=0$ if pendulum doesn't swing over)

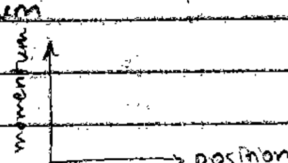
For $\theta = 0$, $p_\theta = 0$:



circles or ellipse corresponds to simple harmonic motion (small oscillations)

$\theta = \pi$, $p_\theta = 0$:

- an impulse is where there is a sudden change in momentum
  ↳ large force in a very small amount of time
  ↳ move up or down in phase space



- The Sudden Approximation
  ↳ change forces instantaneously, momentum changes and follow new set of forces (new "curve" in phase space)

Hillary

- Area enclosed by curves in phase space

$$A = \oint p\,dq = \int_{t_1}^{t_2} p(t)\,\frac{dq(t)}{dt}\,dt$$

- multidimentional system: $A = \oint \sum_i p_i \frac{dq_i}{dt}\,dt = \oint \sum_i p_i \dot{q}_i\,dt$

(eg. an x-p_x plot

(a) y-p_y plot)

t_1 is time where we are at some point in phase space, t_2 is one period (of the motion) later, return to same point in phase space

area enclosed by curve

action

$$\int H\,dt = \int \left(\sum p_i \dot{q}_i - L\right) dt = A - \int L\,dt = A - S$$

if $\frac{\partial H}{\partial t} = 0$: $E(t_2 - t_1) = E \cdot P = A - S$

energy (constant H)

period

A

S

$A = S + E(t_2 - t_1)$ (minimal area equivalent to minimizing action)

$H = H(p, q, \lambda)$ $\lambda = \lambda(t)$ approximation $\boxed{P\frac{d\lambda}{dt} \ll \lambda}$

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}$$

How energy of system changes

$$\overline{\frac{dE}{dt}} = \overline{\frac{\partial H}{\partial \lambda}}\frac{d\lambda}{dt}$$

Hamiltonian changes very slowly, so that H doesn't change by very much in one P.

Adiabatic Changes

11-28-05

$H = H(p,q,\lambda)$; $T\frac{d\lambda}{dt} \ll \lambda$ parameter λ changes very slowly

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} \Rightarrow \frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\frac{\overline{\partial H}}{\partial \lambda}$$

$\overline{x}$ means averaged over a period

$$\frac{\overline{\partial H}}{\partial \lambda} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}\,dt$$

$\frac{\partial H}{\partial t}$ taken for the unperturbed motion

$$T = \int_0^T dt = \oint \frac{dq}{\dot{q}} = \oint \left(\frac{\partial H}{\partial p}\right)^{-1} dq$$

$\dot{q} = \frac{dq}{dt} \Rightarrow dt = \frac{dq}{\dot{q}}$

$$\frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\left[\oint \left(\frac{\partial H}{\partial p}\right)^{-1} dq\right]^{-1} \oint \frac{\partial H}{\partial \lambda}\left(\frac{\partial H}{\partial p}\right)^{-1} dq$$

$$\frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\frac{\overline{\partial H}}{\partial \lambda} = \frac{d\lambda}{dt}\frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}\,dt$$

At any moment in motion (any particular oscillation), momentum is

$p = p(q,E,\lambda)$ a function of position, energy, and λ

$$H(p,q,\lambda) = E \Rightarrow \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0 \quad \therefore \frac{\partial H}{\partial \lambda} = -\frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda}$$

Ex. (Aside) $H = \frac{p^2}{2m} + \frac{1}{2}\lambda x^2 = E \Rightarrow p = \sqrt{2m\left(E - \frac{1}{2}\lambda x^2\right)}$

$$\frac{dH}{d\lambda} = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = \frac{1}{2}x^2 + \left(\frac{p}{m}\right)\left(\frac{-mx^2}{2\sqrt{2m\left(E-\frac{1}{2}\lambda x^2\right)}}\right) = \frac{1}{2}x^2 + \left(\frac{p}{m}\right)\left(\frac{-mx^2}{2p}\right) = 0$$

$$\therefore \frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\left[\oint\left(\frac{\partial H}{\partial p}\right)^{-1} dq\right]^{-1}_1 \oint \left(-\frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda}\right)\left(\frac{\partial H}{\partial p}\right)^{-1} dq$$

$$= -\frac{d\lambda}{dt}\left[\oint \frac{\partial p}{\partial E}\,dq\right]^{-1}\oint \frac{\partial p}{\partial \lambda}\,dq$$

since $\frac{\partial H}{\partial p} = \frac{\partial E}{\partial p} = \left(\frac{\partial p}{\partial E}\right)^{-1}$

b/c q, λ don't have p-dependen[ce]

$$\therefore 0 = \frac{d\overline{E}}{dt}\left[\oint \frac{\partial p}{\partial E}\,dq\right] + \frac{d\lambda}{dt}\left[\oint \frac{\partial p}{\partial \lambda}\,dq\right]$$

$$\therefore\ 0 = \oint \left(\frac{d\bar{E}}{dt} \frac{\partial p}{\partial E} + \frac{d\lambda}{dt} \frac{\partial p}{\partial \lambda} \right) dq = \oint \frac{dp}{dt}\, dq \qquad \text{since } p = p(q, E, t)$$

$$0 = \frac{d}{dt} \underbrace{\oint p\, dq}_{\text{area in phase space}}$$

* Even though shape of trajectory through phase space changes, area enclosed doesn't change → constant

Flow through Phase Space

$dN = f(q_i, p_i)\, \Pi\, dq_i dp_i$ — dN = # of systems at a particular location

f = phase space density

$$\frac{\partial f}{\partial t} + \nabla_q \cdot (f v) + \nabla_p \cdot \left(f \frac{d\vec{p}}{dt} \right) = 0 \Rightarrow \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial}{\partial q_i} (\dot{q}_i f) + \frac{\partial}{\partial p_i} (\dot{p}_i f) \right) = 0$$

$$0 = \frac{\partial f}{\partial t} + \sum_i \left[\frac{\partial}{\partial q_i} \left(\frac{\partial H}{\partial p_i} f \right) + \frac{\partial}{\partial p_i} \left(\left(F_i - \frac{\partial H}{\partial q_i} \right) f \right) \right]$$

$$= \frac{\partial f}{\partial t} + \sum_i \left[\frac{\partial^2 H}{\partial q_i \partial p_i} f + \dot{q}_i \frac{\partial f}{\partial q_i} + \frac{\partial F_i}{\partial p_i} f - \frac{\partial^2 H}{\partial p_i \partial q_i} f - \dot{p}_i \frac{\partial f}{\partial p_i} \right]$$

$$= \frac{\partial f}{\partial t} + \sum_i \left[\dot{q}_i \frac{\partial f}{\partial q_i} + \frac{\partial F_i}{\partial p_i} f - \dot{p}_i \frac{\partial f}{\partial p_i} \right]$$

– When to use Euler's equations? $\frac{d\vec{L}}{dt} = \vec{N}$ – when there are no torques

$$\frac{d\vec{L}}{dt} = \frac{\partial \vec{L}_{body}}{\partial t} + \vec{\omega}_{body} \times \vec{L}_{body}$$

$$\vec{L}_{body} = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix} \qquad \vec{\omega}\times\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \dot{\omega}_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

$$\vec{\omega}\times\vec{L} = \begin{bmatrix} I_3\omega_2\omega_3 - I_2\omega_2\omega_3 \\ I_1\omega_1\omega_3 - I_3\omega_1\omega_3 \\ I_2\omega_1\omega_2 - I_1\omega_1\omega_2 \end{bmatrix}$$

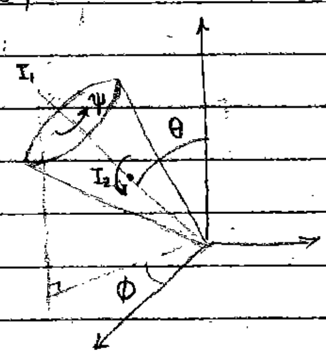
↳ free procession: when $I_1 = I_2$ If $N_1 = N_2 = N_3 = 0$, $\omega_3 = \text{const.}$ $(\dot{\omega}_3 = 0)$

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_1 - I_3) = N_1 \qquad \dot{\omega}_1 - \omega_2\frac{\omega_3(I_1 - I_3)}{I_1} = 0$$
$$I_1\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 = N_3 \qquad \dot{\omega}_2 + \omega_1\frac{\omega_3(I_1 - I_3)}{I_1} = 0$$

$$\therefore \omega_1 = A\sin\left[\frac{\omega_3(I_1 - I_3)t}{I_1}\right] \qquad \omega_2 = A\cos\left[\frac{\omega_3(I_1 - I_3)t}{I_1}\right]$$

– Rigid Body Motion

↳ top:



$$V = mgh = mgl\cos\theta$$
$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2(\omega_2^2 + \omega_3^2) \qquad \text{since } I_2 = I_3$$

$$\omega_1 = \dot{\psi} + \dot{\phi}\cos\theta$$
$$\omega_2 = \dot{\theta}$$
$$\omega_3 = \dot{\phi}\sin\theta$$

when $\theta = \pi/2$, change ϕ doesn't change ψ

$$\therefore L = \frac{1}{2}I_1(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{1}{2}I_2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mgl\cos\theta$$

$$\frac{\partial L}{\partial \psi} = 0 \Rightarrow p_\psi = \frac{\partial L}{\partial \dot{\psi}} \qquad \frac{\partial L}{\partial t} = 0 \rightarrow \text{Hamitonian conserved}$$

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad H = p_\psi\dot{\psi} + p_\phi\dot{\phi} + p_\theta\dot{\theta} - L$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_1(\dot\psi + \dot\phi\cos\theta)$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1(\dot\psi + \dot\phi\cos\theta)\cos\theta + I_2\dot\phi\sin^2\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = I_2\dot\theta$$

Chaotic Motion

$$\Delta p = \frac{\partial p}{\partial \dot{q}} \Delta \dot{q} + \frac{\partial p}{\partial q} \Delta q \quad , \quad p = \frac{\partial L}{\partial \dot{q}} \quad \Rightarrow \quad \Delta p = \frac{\partial^2 L}{\partial \dot{q}^2} \Delta \dot{q} + \frac{\partial^2 L}{\partial q \partial \dot{q}} \Delta q$$

Ex. Double Pendulum