PHYS 306 **Homework 6** Due Monday April 7, *5 PM*

In wall box labeled PHYS 306 outside Henn 310

Late Penalty: -40% by 5 PM Apr 8,

-75% by 5 PM Apr 9.

1. Use the Hamilton method to derive the equations of motion of the rigid-rod pendulum, but whose motion is not confined to lie in the plane. That is, from a fixed pivot point, a massless rod of length *L* is attached to a mass *m* pulle downward by gravity, but is otherwise free to moving anywhere on a sphere of radius *L* centered on the pivoot, using polar angle theta (with $\theta=0$ corresponding to the positive z axis; that is, 'up') and azimuthal angle φ. Pick your potential function (always free to an arbitrary additive constant) so that the potential is zero when $\theta=\pi$.

a) Construct the Lagrangian of the system, and determine the generalized momenta.
b) Build the Hamiltonian via the Legendre transformation, eliminating the time derivatives $\dot{\theta}$ and $\dot{\varphi}$.
c) Derive Hamilton's equations for the system. Are there conserved quatities; if so, which ones?
d) Show that if the system has no momentum in the φ coordinate, you can recover the second order differential equation of motion derived in class for the simple pendulum.
e) Again for the $p_\varphi=0$ case, study the curves of constant values of the Hamiltonian (that is, curves of constant energy, which are thus the trajectories. Here set *m*=1, *L*=1, and *g*=1 (which corresponds to picking particular units of mass, length, and time). What value of H corresponds to no motion (with the mass hanging straight down: $\theta=\pi$, at rest)? What value of H corresponds to the unstable fixed point with the mass directly above the pivot point, and unmoving? (This is unstable, but a fixed point).
f) Sketch a plot (or better yet, draw one using a computer) of θ on the x-axis (from zero to 2π) and *p* on the y-axis, for the two values of H above, as well as one halfway between them, one with twice the energy of the unstable solution. [Note that for every value of θ there are TWO values of *p*, and that you must take care to note that for some values of H certain values of θ are forbidden.] These curves are the trajectories; we solved for the time behaviour *along* these trajectories in lecture, in terms of Jacobian elliptic functions.

2. (a) Prove that the transformation :

$$P=\frac{1}{2}\left(p^2+q^2\right)$$
$$Q=\arctan\left(q/p\right)$$

is canonical. There are a variety of methods to do this.

(b) Take the simple harmonic osciallator Hamiltonian derived in tutoral, and use units where the mass is unity and the spring constant k=1 also. What is the Hamiltonian K(P,Q)? Write down Hamilton's equations for *P* and *Q*, and solve them by explicit integration. Note that for an initial energy E, *K(P,Q)* is time independent, so *K*=E always; with this identification, invert the coordination transformation in order to obtian *q(t)* and *p(t),* interpreting the constants that came out of the earlier integration.