Solutions to HW2

Problem 1

Let's start with the expression for the velocity in spherical-polar coordinates:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} \tag{1}$$

where the $\hat{r}, \hat{\theta}, \hat{\phi}$ (unit) vectors can be put in terms of the Cartesian unit vectors, as shown in a previous tutorial:

$$\begin{aligned}
\hat{r} &= \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \\
\hat{\theta} &= \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi - \sin\theta\hat{k} \\
\hat{\phi} &= -\sin\phi\hat{i} + \cos\phi\hat{j}
\end{aligned} \tag{2}$$

In order to find the acceleration, we need to take the time derivative of the velocity; we can apply the derivative of Equation 1 and apply chain rule to get the acceleration:

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}\right) \\
&= \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} + \dot{r}\sin\theta\dot{\phi}\hat{\phi} + r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi} + r\sin\theta\ddot{\phi} + r\sin\theta\dot{\phi}\dot{\hat{\phi}}
\end{aligned} \tag{3}$$

In order to proceed, we need to find the time derivatives of each spherical-polar unit vector. So let's consider the $\hat{r}$ vector first, by applying change rule for each of the three variables:

$$\begin{aligned}
\frac{d}{dt}\hat{r} &= \frac{dr}{dt}\frac{\partial\hat{r}}{\partial r} + \frac{d\theta}{dt}\frac{\partial\hat{r}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{r}}{\partial\phi} \\
&= \dot{r}\big(0\big) + \dot{\theta}\big(\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta k\big) + \dot{\phi}\big(-\sin\theta\sin\phi\hat{i} + \sin\theta\cos\phi\hat{j}\big) \\
&= \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}
\end{aligned} \tag{4}$$

We can find the relations for the other time derivatives of unit vectors using the same method:

$$
\begin{aligned}
\frac{d}{dt}\hat{\theta} &= \frac{dr}{dt}\frac{\partial\hat{\theta}}{\partial r} + \frac{d\theta}{dt}\frac{\partial\hat{\theta}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{\theta}}{\partial\phi} \\
&= \dot{r}\big(0\big) + \dot{\theta}\big(- \sin\theta\cos\phi\hat{i} - \sin\theta\sin\phi\hat{j} - \cos\theta k\big) + \dot{\phi}\big(- \cos\theta\sin\phi\hat{i} + \cos\theta\cos\phi\hat{j}\big) \\
&= -\dot{\theta}\hat{r} + \cos\theta\dot{\phi}\hat{\phi} && (5) \\
\frac{d}{dt}\hat{\phi} &= \frac{dr}{dt}\frac{\partial\hat{\phi}}{\partial r} + \frac{d\theta}{dt}\frac{\partial\hat{\phi}}{\partial\theta} + \frac{d\phi}{dt}\frac{\partial\hat{\phi}}{\partial\phi} \\
&= \dot{r}\big(0\big) + \dot{\theta}\big(0\big) + \dot{\phi}\big(- \cos\phi\hat{i} - \sin\phi\hat{j}\big) \\
&= \dot{\phi}\Big[- (\sin^2\theta + \cos^2\theta)\cos\phi\hat{i} - (\sin^2\theta + \cos^2\theta)\sin\phi\hat{j}\Big] \\
&= \dot{\phi}\Big[- (\sin^2\theta + \cos^2\theta)\cos\phi\hat{i} - (\sin^2\theta + \cos^2\theta)\sin\phi\hat{j}\Big] \\
&= \dot{\phi}\Big[- (\sin^2\theta\cos\phi\hat{i} + \sin^2\theta\sin\phi\hat{j}) - (\cos^2\theta\cos\phi\hat{i} + \cos^2\theta\sin\phi\hat{j})\Big] \\
&= \dot{\phi}\Big[- \sin\theta(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j}) - \cos\theta(\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j})\Big] \\
&= - \sin\theta\dot{\phi}\hat{r} - \cos\theta\dot{\phi}\hat{\theta} && (6)
\end{aligned}
$$

We can now take Equations 4, 6 and 6, and plug them into Equation 3 to get an expression of the acceleration in terms of the spherical-polar coordinates and their unit vectors:

$$
\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}\right) \\
&= \ddot{r}\hat{r} + \dot{r}(\dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}) + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta}\hat{r} + \cos\theta\dot{\phi}\hat{\phi}) + \dot{r}\sin\theta\dot{\phi}\hat{\phi} \\
&\quad + r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi} + r\sin\theta\ddot{\phi} + r\sin\theta\dot{\phi}(-\sin\theta\dot{\phi}\hat{r} - \cos\theta\dot{\phi}\hat{\theta}) \\
&= \hat{r}(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2) + \hat{\theta}(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2) + \hat{\phi}(r\sin\theta\ddot{\phi} + 2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi})
\end{aligned}
$$

Problem 2

Let's start with the usual expression for kinetic energy:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{7}$$

We can express x, y and z as functions of the radius r and angles θ, ϕ using the spherical-polar transformations:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{8}$$

We can then take time derivatives of each coordinate, and plug into Equation 7. In the case of the time derivative of x:

$$\begin{aligned} \dot{x}^2 &= (\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi)^2 \\ &= \dot{r}^2\sin^2\theta\cos^2\phi + r^2\dot{\theta}^2\cos^2\theta\cos^2\phi - r^2\dot{\phi}^2\sin^2\theta\sin^2\phi \\ &\quad + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\cos^2\phi - 2r\dot{r}\dot{\phi}\sin^2\theta\sin\phi\cos\phi - 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi \end{aligned} \tag{9}$$

The y derivative can be found using the same method:

$$\begin{aligned} \dot{y}^2 &= (\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi)^2 \\ &= \dot{r}^2\sin^2\theta\sin^2\phi + r^2\dot{\theta}^2\cos^2\theta\sin^2\phi + r^2\dot{\phi}^2\sin^2\theta\cos^2\phi \\ &\quad + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\sin^2\phi + 2r\dot{r}\dot{\phi}\sin^2\theta\sin\phi\cos\phi + 2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi \end{aligned} \tag{10}$$

and the z time derivative is simpler to compute:

$$\begin{aligned} \dot{z}^2 &= (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 \\ &= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\sin\theta\cos\theta \end{aligned} \tag{11}$$

When Equations 9, 10 and 11 are added together, many cross terms immediately cancel, and some eventually cancel when you note that $\sin^2\phi + \cos^2\phi = 1$. The remaining terms can be simplified further when noting that $\sin^2\theta + \cos^2\theta = 1$; the kinetic energy can therefore be expressed in terms of spherical-polar parameters and their time derivatives:

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) \end{aligned} \tag{12}$$

Problem 3

The electrostatic force acting between two charges q_1 and q_2 is $F = kq_1q_2/r^2$, where r is the distance between the two particles and is a constant. The electrostatic potential energy is defined to be

$$U = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} = \frac{kq_1q_2}{r}$$

in this problem, $q_1 = e$ is the charge of the incoming proton, and $q_2 = Ze$ is the total charge of the protons that make up the nucleus. Therefore, the form of the Lagrangian for this system is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - \frac{Ze^2}{r} \end{aligned} \tag{13}$$

There are three possible Euler-Lagrange equations, since there are three coordinates (r, θ, ϕ). In the case of r:

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) &= 0 \\ &= \frac{Ze^2}{r^2} + mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2 - m\ddot{r} \end{aligned} \tag{14}$$

In the case of θ:

$$\begin{aligned} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 0 \\ &= mr^2\sin\theta\cos\theta\dot{\phi}^2 - \frac{d}{dt}\big(mr^2\dot{\theta}\big) \end{aligned} \tag{15}$$

In the case of ϕ:

$$\begin{aligned} \frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= 0 \\ &= 0 - \frac{d}{dt}\big(mr^2\sin^2\theta\dot{\phi}\big) \end{aligned} \tag{16}$$

The ϕ equation states that the quantity $mr^2\sin^2\theta\dot{\phi} = \text{constant} = h$. The time derivative of h requires that

$$\begin{aligned} \frac{dh}{dt} &= 2mr\dot{r}\sin^2\theta\dot{\phi} + 2mr^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + mr^2\sin^2\theta\ddot{\phi} \\ &= 0 \end{aligned} \tag{17}$$

According to Equation 17, $\ddot{\phi} = 0$ if, at some instance, $\dot{\phi} = 0$; the value of ϕ is therefore conserved in this case! We can then simplify Equations 14 and 15 since $\dot{\phi} = 0$:

$$
\begin{aligned}
m\ddot{r} &= \frac{Ze^2}{r^2} + mrr\dot{\theta}^2 \\
\frac{d}{dt}\left(mr^2\theta\right) &= 0 \\
\phi &= constant
\end{aligned}
\tag{18}
$$