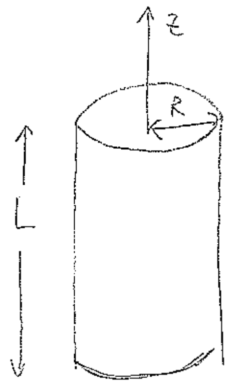
① Moment of Inertia of a cylinder



mass M

$$I_z = \frac{M}{V}\int d^3r\,(x^2+y^2) \qquad ;\ V = \pi R^2 L$$

$$= \frac{M}{V}\int_0^{2\pi} d\varphi \int_{-L/2}^{L/2} dz \int_0^R dr\, r\,(r^2\cos^2\varphi + r^2\sin^2\varphi)$$

$$= \frac{M}{V}\, 2\pi L \int_0^R dr\; r^3$$

$$= \frac{M}{V}\, 2\pi L\, \frac{1}{4} r^4 \Big|_0^R = \frac{M}{\pi R^2 L}\,\frac{\pi L}{2} R^4 = \underline{\underline{\frac{M}{2} R^2}}$$

$$I_x = \frac{M}{V}\int d^3r\,(y^2+z^2)$$

$$= \frac{M}{V}\int_0^{2\pi} d\varphi \int_{-L/2}^{L/2} dz \int_0^R dr\, r\,(r^2\sin^2\varphi + z^2)$$

NB: $\int d\varphi \sin^2\varphi = -\cos\varphi\sin\varphi + \int d\varphi\cos^2\varphi$

$= -\cos\varphi\sin\varphi + \int d\varphi - \int d\varphi\sin^2\varphi$

$\Rightarrow \int d\varphi \sin^2\varphi = \dfrac{\varphi - \cos\varphi\sin\varphi}{2}$

$\Rightarrow \int_0^{2\pi} d\varphi \sin^2\varphi = \pi$

Therefore:

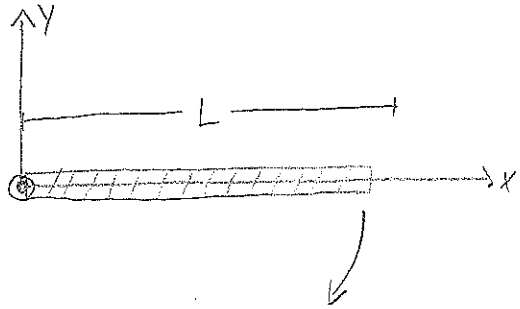
$$I_x = \frac{M}{V}\left[\pi L \frac{1}{4} R^4 + 2\pi \left(\frac{1}{3} z^3 \Big|_{-L/2}^{L/2}\right) \frac{1}{2} R^2\right]$$

$$= \frac{M}{V}\left(\frac{\pi R^4 L}{4} + \frac{\pi L^3 R^2}{12}\right) \quad \text{and } V = \pi R^2 L$$

$$= \underline{\underline{\frac{M}{12}\left(3R^2 + L^2\right)}}$$

Symmetry: $I_Y = I_X$.

(2)



$$I_z = \frac{\mu}{L} \int_0^L dx\, x^2$$

$$= \frac{1}{3} \mu L^2$$

a) conservation of energy:

initial : $E_{pot} = 0$; $E_{rot} = 0$

final : $E_{pot} = -\mu g \frac{L}{2}$; $E_{rot} = \frac{1}{2} I_z \omega^2$

$$\Rightarrow \quad \frac{1}{2} I_z \omega^2 - \mu g \frac{L}{2} = 0$$

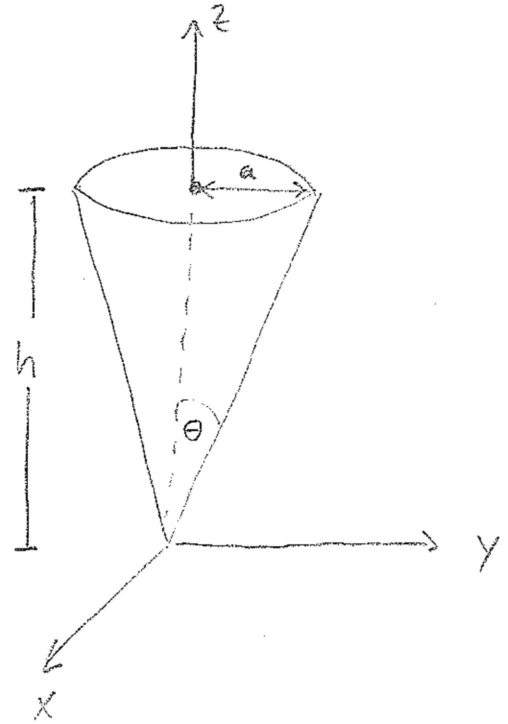
$$\Leftrightarrow \quad \omega = \sqrt{\frac{\mu g L}{I_z}}$$

$$\Leftrightarrow \quad \underline{\underline{\omega = \sqrt{\frac{3g}{L}}}}$$

b) $V_{cm} = \frac{L}{2} \omega = \sqrt{\frac{3gL}{4}}$

lowest point: $V_{LP} = L\omega = \sqrt{3gL}$

(3)



Θ : opening angle

$\tan\Theta = \frac{a}{h}$

$$I_z = \frac{M}{V} \int d^3r \; (x^2+y^2) \quad ; \quad V = \frac{1}{3}\pi a^2 h$$

$$= \frac{M}{V} \int_0^{2\pi} d\varphi \int_0^h dz \int_0^{z\tan\Theta} dr \; r\,(r^2)$$

$$= \frac{M}{V}\, 2\pi \int_0^h dz \; \frac{(z\tan\Theta)^4}{4}$$

$$= \frac{M}{V}\, \frac{\pi}{10} (\tan\Theta)^4 \, h^5$$

$$= \frac{3M\,\pi\,(a/h)^4 h^5}{10\,\pi a^2 h}$$

$$= \frac{3}{10} M a^2$$

I_z is the same for axis through vertex and axis through CM

2P

for axes through vertex:

$$I_x' = \frac{\mu}{V} \int d^3r \, (y^2+z^2)$$

$$I_y' = \frac{\mu}{V} \int d^3r \, (x^2+z^2)$$

$$\Rightarrow \quad I_x' + I_y' = \frac{\mu}{V} \int d^3r \, (x^2+y^2 + 2z^2)$$

$$= I_z + \frac{2\mu}{V} \int d^3r \, z^2$$

$$= \frac{2\mu}{V} \int_0^{2\pi} d\varphi \int_0^{h} dz \int_0^{z \tan\theta} dr \, r \, z^2$$

$$= \frac{2\mu}{V} \, 2\pi \int_0^h dz \, \frac{1}{2} \tan^2\theta \, z^2 \cdot z^2$$

$$= \frac{2\pi\mu}{V} \tan^2\theta \, \frac{1}{5} h^5$$

$$= \frac{2\pi\mu}{5V} a^2 h^3$$

$$= \frac{3 \cdot 2\pi \mu \, a^2 h^3}{5 \, \pi a^2 h}$$

$$= \frac{6}{5} \mu h^2$$

Symmetry: $I_x' = I_y' \Rightarrow I_x' = \frac{1}{2} I_z + \frac{\mu}{V} \int d^3 z^2$

$$= \frac{3}{20} \mu a^2 + \frac{3}{5} \mu h^2$$

$$= \underline{\underline{\frac{3}{20} \mu (a^2 + 4h^2)}} \qquad 4P$$

(3) continued:

position of CM:

$$\vec{R}_{CM} = R_{CM}\,\hat{e}_z \qquad \text{by symmetry}$$

$$R_{CM} = \frac{1}{V}\int d^3r \; \hat{=}$$

$$= \frac{1}{V}\int_0^{2\pi} d\varphi \int_0^h dz \int_0^{z\tan\theta} dr\; r\, z$$

$$= \frac{1}{V}\, 2\pi \int_0^h dz \; \frac{1}{2}\left(z^2 \tan^2\theta\right) z$$

$$= \frac{1}{V}\, \pi \tan^2\theta \; \frac{1}{4}\, h^4$$

$$= \frac{1}{V}\, \pi/3 \; a^2 h^2$$

$$= \frac{3 \quad \pi\, a^2 h^2}{4\ \pi a^2 h}$$

$$= \underline{\underline{\frac{3}{4}\, h}} \qquad 2p$$

Moments of Inertia for axes through CM:

$$I_x = I_x' - \mu R_{CM}^2$$

$$= \frac{3}{20}\mu(a^2 + 4h^2) - \mu \frac{9}{16} h^2$$

$$= \frac{3}{20}\mu a^2 + \left(\frac{3}{5} - \frac{9}{16}\right)\mu h^2$$

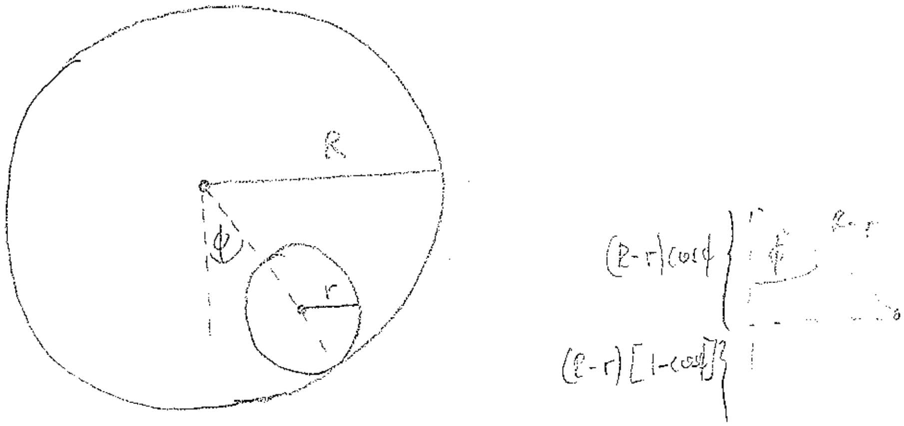
$$= \frac{3}{20}\mu a^2 + \frac{3}{80}\mu h^2$$

$$= \frac{3}{20}\mu\left(a^2 + \frac{1}{4}h^2\right)$$

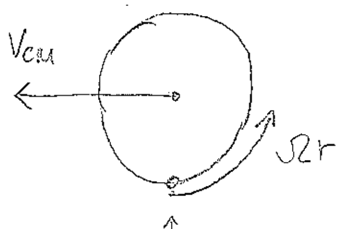
same for I_y.

2p

④



rolling without slipping:



at this point V_{CM} and Ωr must cancel.

$\Rightarrow \quad V_{CM} = \Omega r$

conservation of energy:

$$\frac{1}{2}\mu V_{CM}^2 + \frac{1}{2} I_z \Omega^2 = \mu g (R-r)[1-\cos\phi]$$

$$\Leftrightarrow \quad \frac{1}{2}(\mu r^2 + I_z)\Omega^2 = \mu g (R-r)[1-\cos\phi] \quad \text{and} \quad I_z = \frac{1}{2}\mu r^2$$

$$\Leftrightarrow \quad \Omega = \sqrt{\frac{4g(R-r)[1-\cos\phi]}{3r^2}}$$