Solutions

Phys 306 Midterm

2014

1) a) 1 degree of freedom $q_1 = \theta$

$$V(\theta) = 0$$

$$T(\theta,\dot\theta) = \frac{1}{2} m \vec{v}\cdot\vec{v} = \frac{1}{2} m(\dot x^2 + \dot y^2)$$

$$x(t) = b\cos(\omega t) + b\cos(\omega t + \theta)$$

$$y(t) = b\sin(\omega t) + b\sin(\omega t + \theta)$$

$$\dot x(t) = -b\omega\sin(\omega t) - b\sin(\omega t + \theta)(\omega + \dot\theta)$$

$$\dot y(t) = b\left(\omega\cos(\omega t) + (\omega + \dot\theta)\cos(\omega t + \theta)\right)$$

$$\mathcal{L}(\theta,\dot\theta,t) = T(\theta,\dot\theta) = \frac{1}{2}m\left(b^2\omega^2 + b^2(\omega+\dot\theta)^2 + 2b^2\omega(\omega+\dot\theta)\left(\sin(\omega t)\sin(\omega t+\theta) + \cos(\omega t)\cos(\omega t+\theta)\right)\right)$$

$$= \frac{1}{2}mb^2\left(2\omega^2 + 2\omega\dot\theta + \dot\theta^2 + 2b^2\omega(\omega+\dot\theta)\cos\theta\right)$$

$$\mathcal{L}(\theta,\dot\theta,t) = \frac{1}{2}mb^2\left(2\omega^2 + 2\omega\dot\theta + \dot\theta^2 + 2\omega(\omega+\dot\theta)\cos\theta\right)$$

$$\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) = 0$$

$$-\cancel{\tfrac{1}{2}mb^2}\left(2\omega(\omega+\dot\theta)\sin\theta\right) - \cancel{\tfrac{1}{2}mb^2}\frac{d}{dt}\left(\cancel{2\omega} + 2\dot\theta + 2\omega\cos\theta\right) = 0$$

$$-\omega(\omega+\dot\theta)\sin\theta - \ddot\theta + \omega(\sin\theta)\dot\theta = 0$$

b) $\boxed{-\omega^2\sin\theta = \ddot\theta}$

c) $T = \frac{2\pi}{\omega}$ for θ small

2a) $E = T + V$

$V(x) = -T(x)$

$-\frac{1}{2}m\dot{x}^2 = -\frac{m\omega^2}{2}(a^2 - x^2)$

$\Rightarrow \dot{x}^2 = \omega^2(a^2 - x^2)$

$\dot{x} = \sqrt{\omega^2(a^2-x^2)}$

$\frac{d\dot{x}}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{\omega^2(a^2-x^2)}} = 0 \Rightarrow x = 0$ $\quad x < a$

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$\ddot{x}$ max when $x = 0$

$\dot{x} = \pm\omega a$

b) Total Energy constant and positive
velocity will initially be $\dot{x} = \pm\omega a + C_1$
where C_1 is a ~~constant~~ $\sqrt{E_0/m}$
and will keep decreasing in speed until $x = a$
where speed $\dot{x} = C_1 = \sqrt{\frac{2E_0}{m}}$ as it will then continue at constant
speed $\dot{x} = \sqrt{\frac{2E_0}{m}}$ as $|x|$ increases

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c) $\frac{dx}{dt} = \sqrt{\omega^2(a^2-x^2)} = \omega\sqrt{(a^2-x^2)}$

$\Rightarrow \int_0^t dt = \frac{1}{\omega}\int \frac{dx'}{\sqrt{a^2-x^2}} = \frac{\arcsin\left(\frac{x'}{a}\right)\Big]_{-a}^{x}}{\omega} \Rightarrow \frac{-\sin^{-1}(-1) + \sin^{-1}\left(\frac{x}{a}\right)}{\omega} = t$

$\underline{t(x) = +\left(\pi/2 + \arcsin\left(x/a\right)\right)/\omega}$

for $X = a$ $\quad t(a) = +\left(\pi/2 + \arcsin(1)\right)/\omega = \underline{\pi/\omega}$

Excellent!

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3a) $I_x = \int (y^2+z^2)\,dm = \frac{\mu}{V}\int y^2 dy$ $\quad z=0$

$I_y = \int (x^2+z^2)\,dm = \frac{\mu}{V}\int x^2 dx$

$I_z = \int (x^2+y^2)\,dm = \frac{\mu}{V}\int_0^{2\pi}\int_0^{R} r^2 r\,dr\,d\theta \quad x^2+y^2=r$

$\frac{\mu}{V}\int_0^{2\pi}\int_0^{R} r^3\,dr\,d\theta$

$V = \pi R^2 \qquad \frac{\pi R^4}{2}\cdot\frac{m}{\pi R^2} = \frac{mR^2}{2}$

$I_x = I_y$ by symmetry

$I_z = I_x + I_y = 2I_x \Rightarrow I_y = \frac{mR^2}{4} = I_y$

$\underline{I_z = \frac{mR^2}{2}}$
$I_x = \frac{mR^2}{4}$
$I_y = \frac{mR^2}{4}$

3b) Euler's Equations relate the rotation vector $\vec{\omega}$ to the torque $\vec{\tau}$

$$I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 = \tau_1$$
$$I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_1\omega_3 = \tau_2$$
$$I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 = \tau_3$$

In this problem $\dot{\vec{\omega}} = 0$, $\omega_1 = \omega_x = \omega\sin\theta$
$\omega_2 = \omega_y = 0$
$\omega_3 = \omega_z = \omega\cos\theta$

and $I_1 = I_2 = \frac{1}{4}mR^2$, $I_3 = \cancel{I_z} = \frac{1}{2}mR^2$

$$\therefore \begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\end{pmatrix} = \begin{pmatrix}0\\ -\frac{1}{4}mR^2\omega^2\sin\theta\cos\theta\\ 0\end{pmatrix}$$