

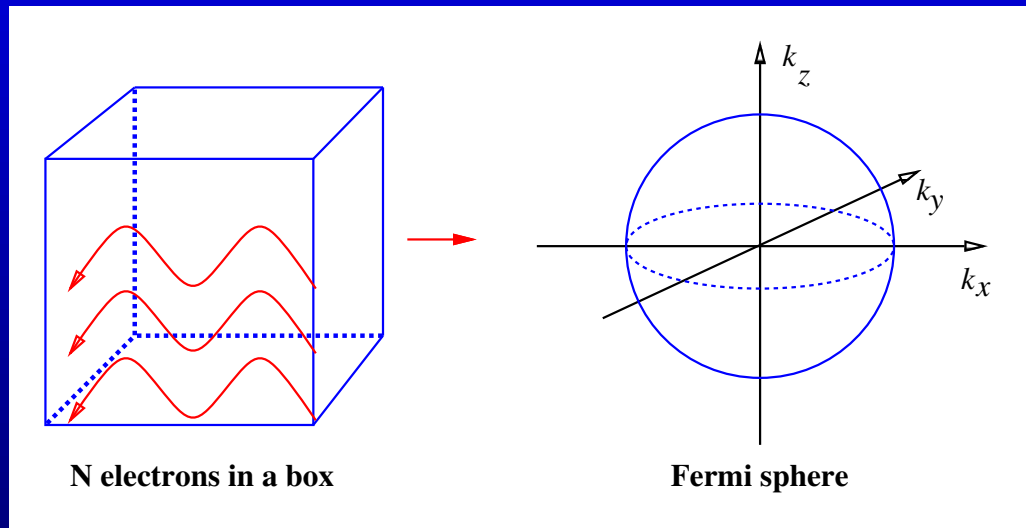
Landau's Fermi liquid paradigm



Lev Davidovich Landau:

“Electron states in solids are adiabatically connected to the states of noninteracting electron gas”

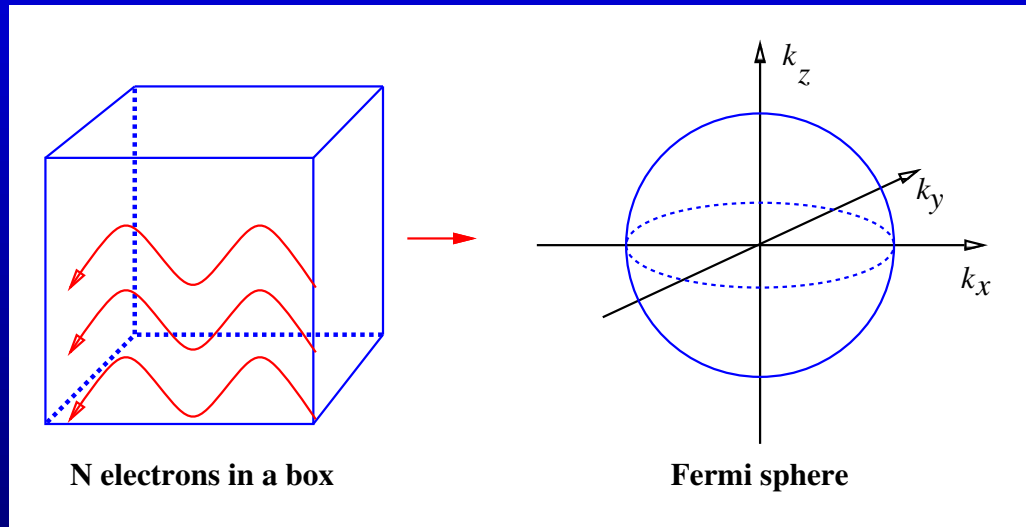
Despite enormous Coulomb forces ($U_C \sim \frac{e^2}{a_0} \sim 1 - 10\text{eV}$) at low energies most metals behave like a **free electron gas** [Landau, 1957]



$$\epsilon(\mathbf{k}) = \frac{\hbar^2 \mathbf{k}^2}{2m^*}, \quad k_F = (3\pi^2 n)^{1/3}$$

Ground state ($T = 0$): all levels below Fermi momentum k_F are filled; levels above k_F empty.

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Root cause: **Pauli exclusion principle**

→ **phase space for scattering near FS is severely limited.**

Structure of electron propagator in FL

$$G(\mathbf{k}, \omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} - \Sigma(\mathbf{k}, \omega)} = \frac{z_{\mathbf{k}}}{\omega - E_{\mathbf{k}} + i\Gamma_{\mathbf{k}}} + G_{\text{incoh}}(\mathbf{k}, \omega)$$

with

$$z_{\mathbf{k}}^{-1} = \left[1 - \frac{\partial \text{Re}\Sigma}{\partial \omega}\right]_{\omega=E_{\mathbf{k}}}, \text{ "quasiparticle weight"}$$

$$\tau^{-1} \equiv \Gamma_{\mathbf{k}} \sim (E_{\mathbf{k}} - E_F)^2, \text{ "quasiparticle lifetime"}$$

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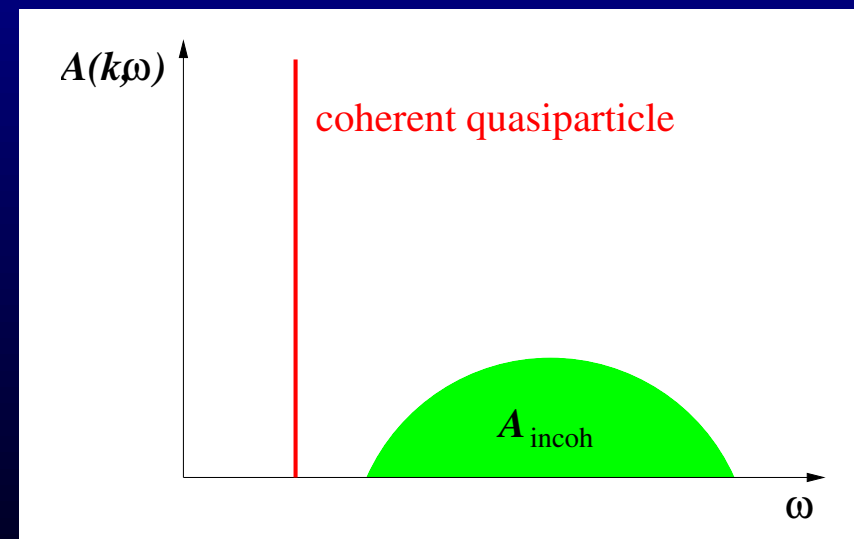
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Spectral function:

$$\begin{aligned} A(\mathbf{k}, \omega) &= -2\text{Im}G(\mathbf{k}, \omega) \\ &\simeq z_{\mathbf{k}}\delta(\omega - E_{\mathbf{k}}) + A_{\text{incoh}}(\mathbf{k}, \omega) \end{aligned}$$



Exceptions to FL paradigm

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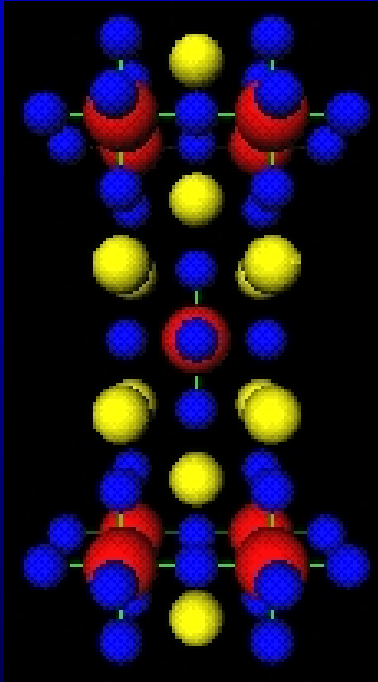
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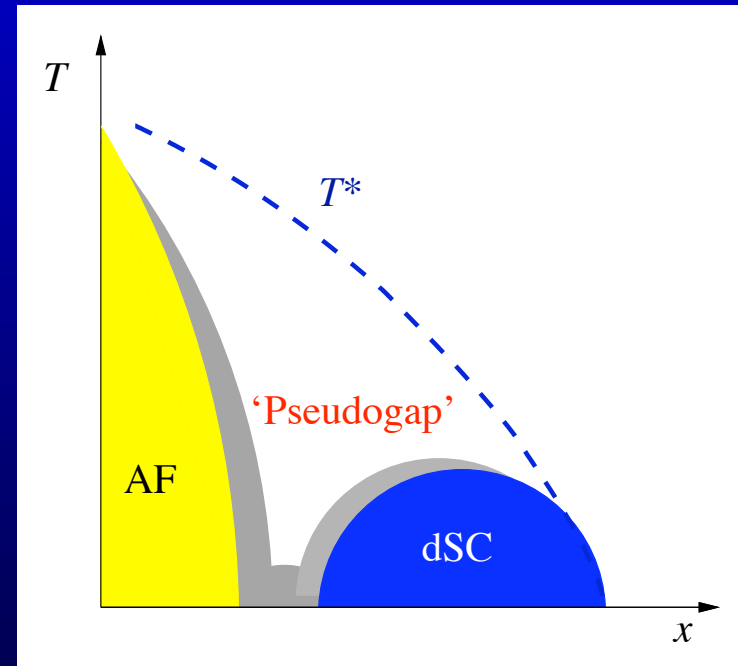
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- Quantum Hall Fluids
- Systems near Quantum Criticality
- High- T_c Cuprate Superconductors (?)

Cuprate superconductors

Cuprates are layered quasi-2D materials with electronic properties dominated by the CuO_2 layers.



Crystal structure of $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$



Phase diagram of cuprates.

d-wave superconductivity in cuprates

Superconducting order parameter in cuprates exhibits *d*-wave symmetry

$$\Delta_{\mathbf{k}} = \frac{1}{2}\Delta_0(\cos k_x - \cos k_y),$$

i.e. changes sign upon 90° rotation.

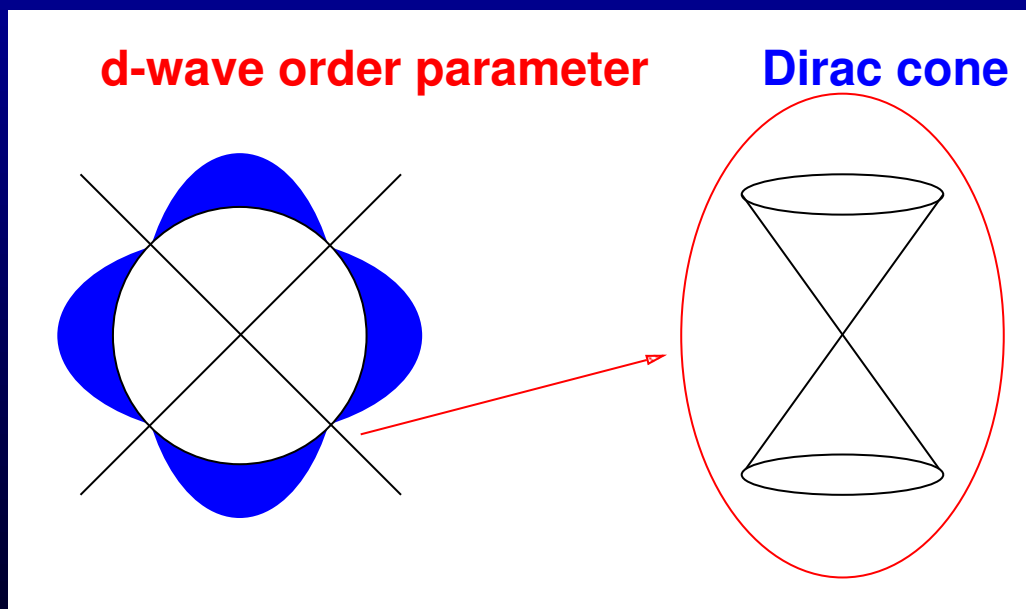
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“Dirac Fermions”

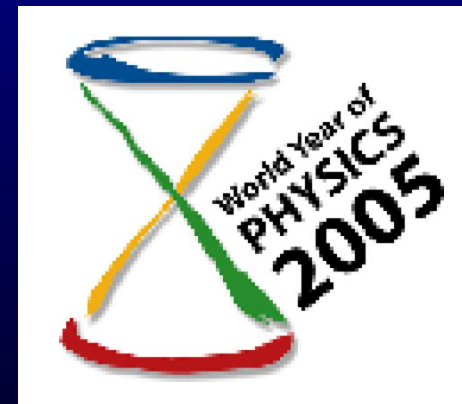
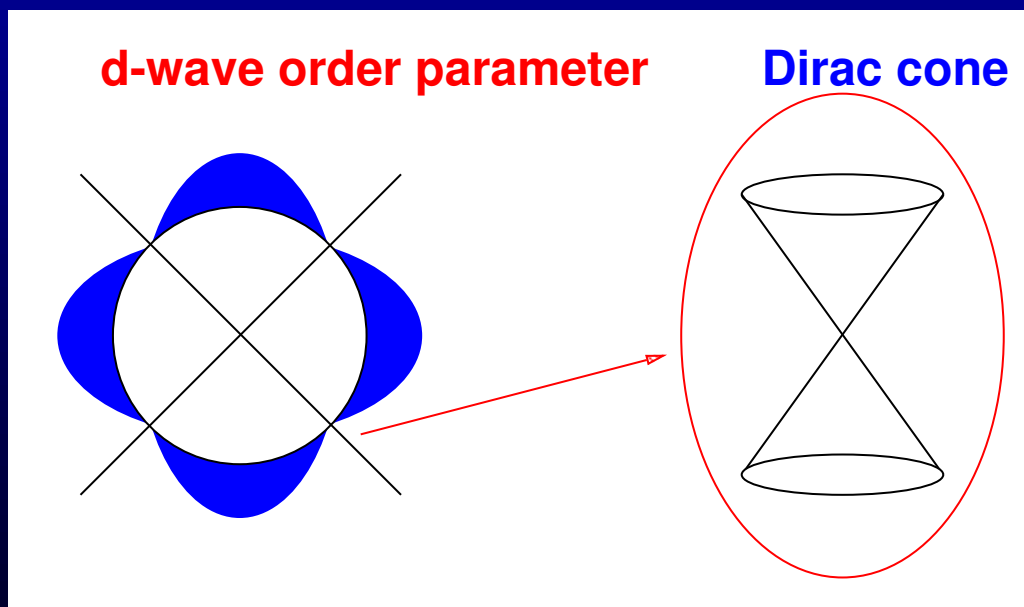
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Superfluid density

Superfluid density ρ_s is a fundamental characteristic of a superconductor reflecting its ability to **carry supercurrent** in response to applied magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$:

$$\mathbf{j}_s = \rho_s \frac{4e^2}{\hbar^2 c} \mathbf{A}.$$

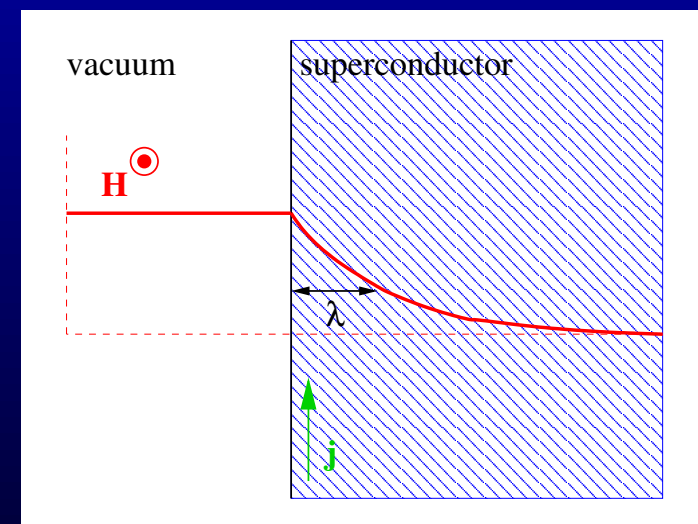
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ρ_s is related to London penetration depth λ , a fundamental lengthscale in a superconductor describing penetration of magnetic field \mathbf{H} into the bulk via

$$\rho_s = \frac{\hbar^2 c^2}{16\pi e^2 \lambda^2}$$

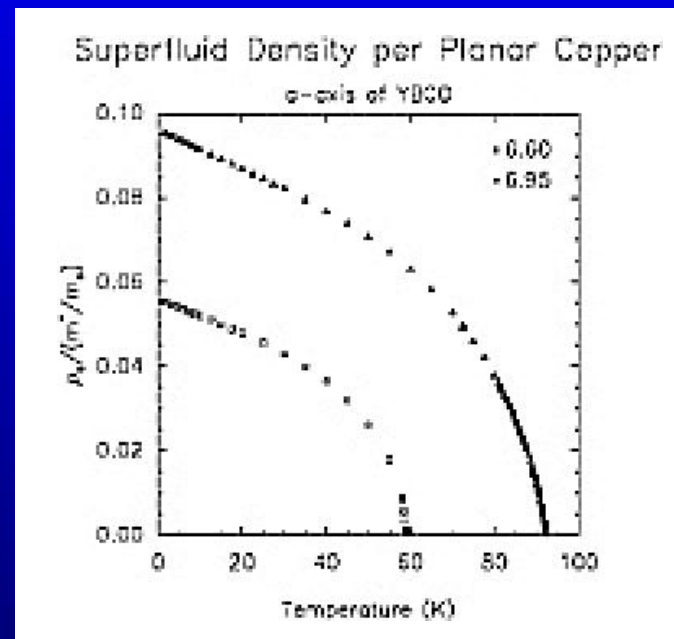


Superfluid density in cuprates, *ab*-plane: the old story

In the optimally doped and moderately underdoped region experiments show

$$\rho_s^{ab}(x, T) \sim \lambda_{ab}^{-2}(x, T) \simeq ax - bk_B T,$$

with $a \simeq 244\text{meV}$ and $b \simeq 3.0$ [Lee and Wen, PRL **78**, 4111 (1997)].

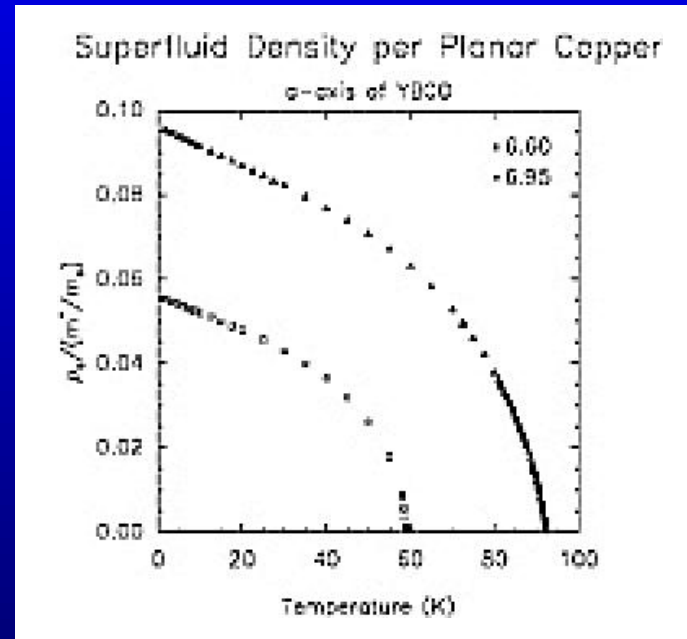


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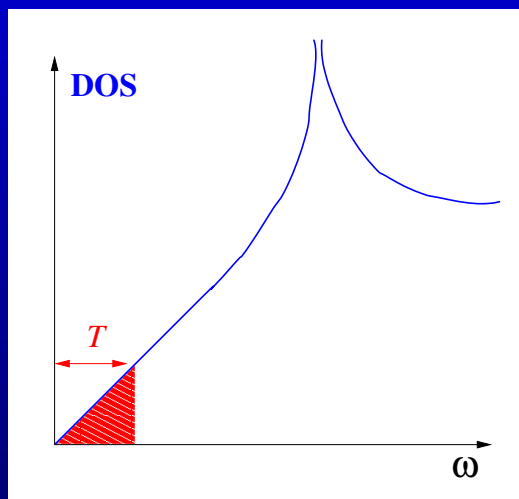
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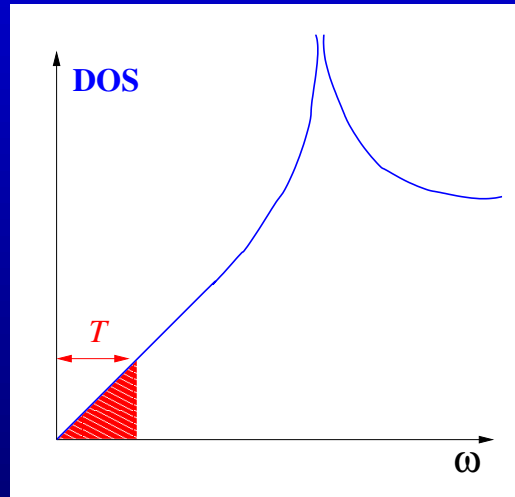
In a BCS *d*-wave superconductor one would have

$$\rho_s^{ab}(x, T) \simeq a(1 - x) - 2 \ln 2 \frac{v_F}{v_\Delta} k_B T.$$

- The linear T -dependence is known to arise from thermally excited **nodal quasiparticles** which exhibit *linear* density of states at low energies.

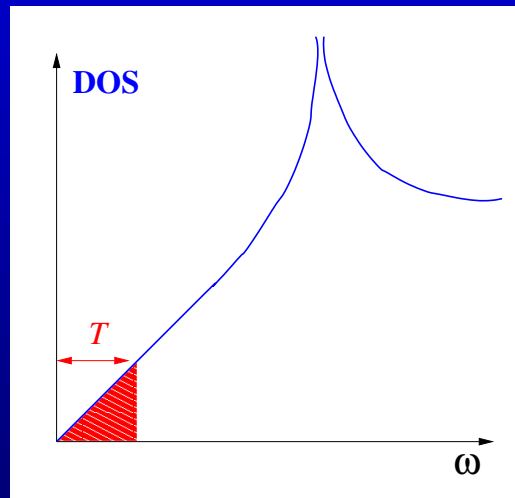


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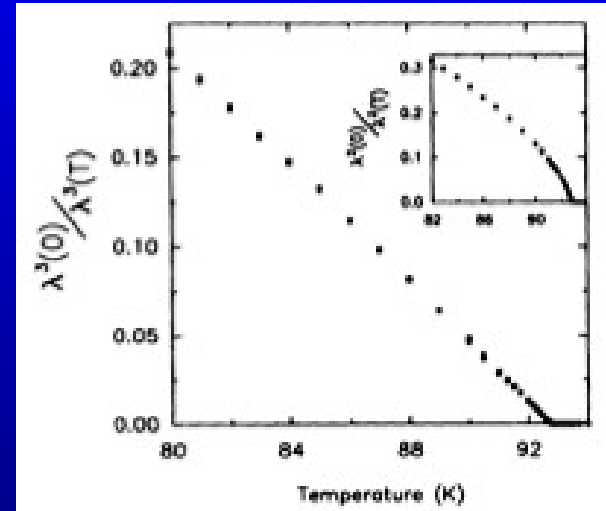


- The linear x -dependence (cf. Uemura plot) reflects proximity to the Mott-Hubbard insulator at half filling.
- **Problem:** models that give correct x -dependence (e.g. RVB-type theories) generally yield strong ($\sim x^2$) dependence of the coefficient b .

3D-XY critical scaling

Optimally doped YBCO shows 3D-XY critical behavior near T_c : $\rho_s(T) \sim (T_c - T)^{2/3}$ with relatively wide critical region $\Delta T \simeq 10\text{K}$.

Such critical behavior is characteristic of **phase disordering** transition to the normal state. In 3d this is known to be caused by **vortex loop unbinding**.

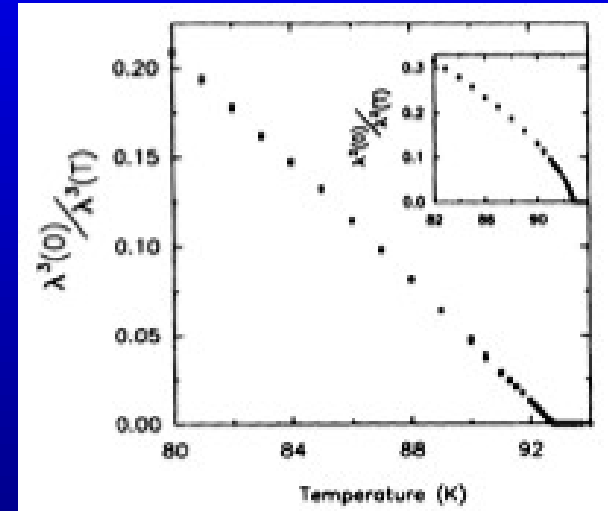


Kamal et al., PRL **73** 1845 (1994)

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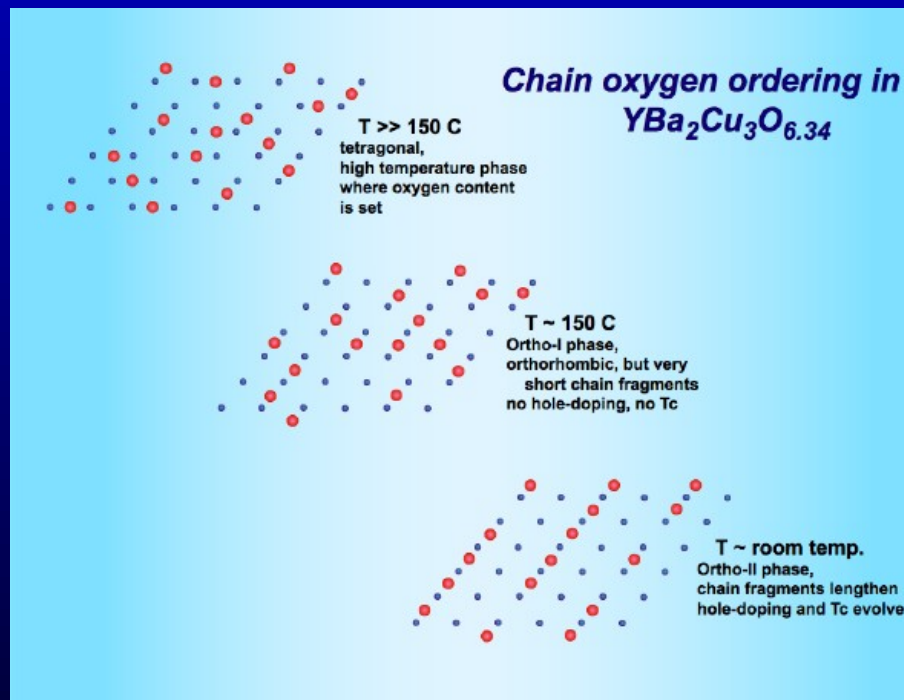
3D-XY critical behavior has also been observed in other high- T_c compounds using transport and thermodynamic probes. It thus appears to be a **fundamental universal** feature of cuprates.

Superfluid density in cuprates: the new story

Recent UBC Group data on ultra-pure YBCO single crystals for doping levels as low as $T_c = 5\text{K}$ show *ab*-plane show tantalizing *qualitative deviations* from the “old” phenomenology.

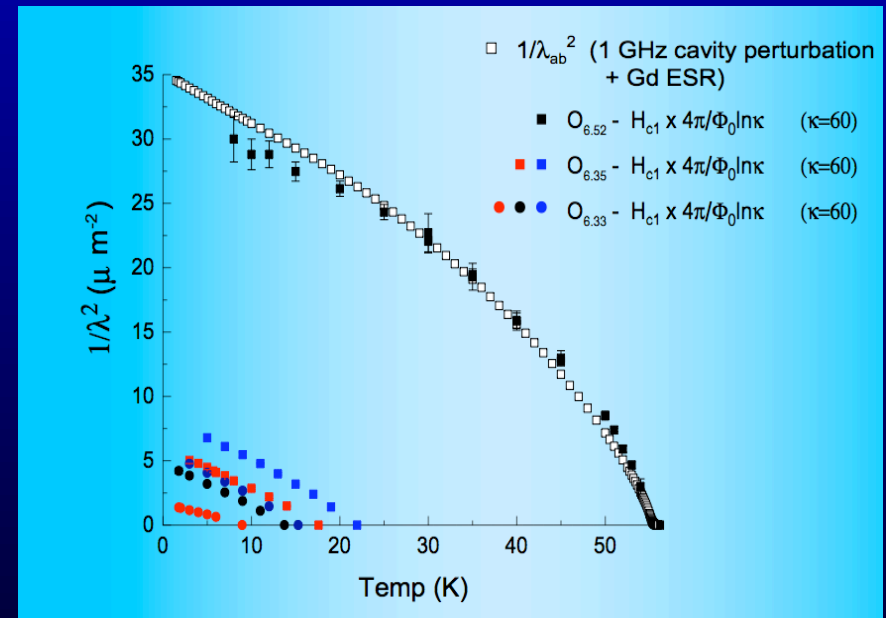
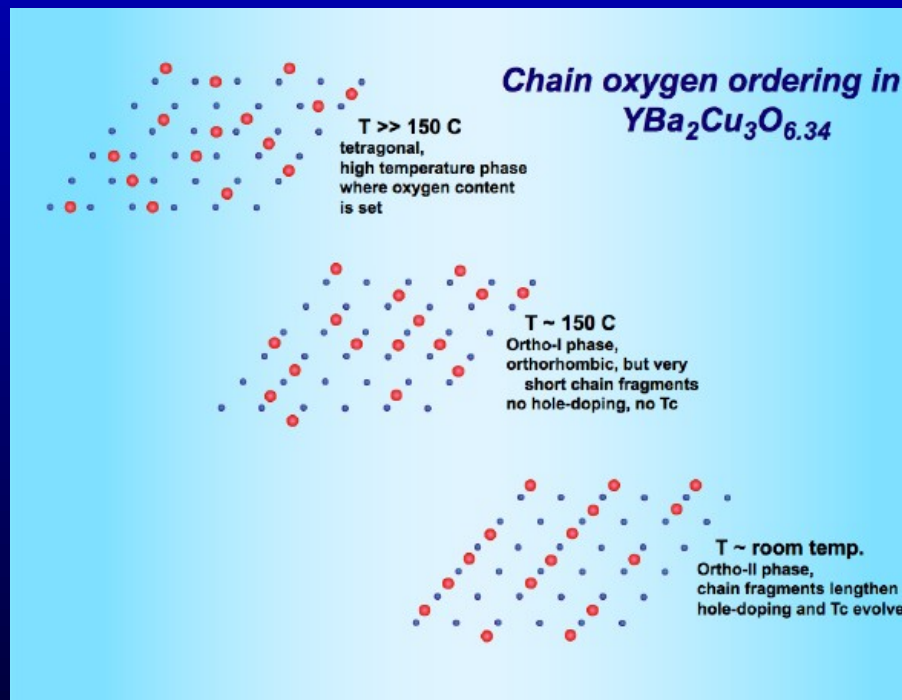
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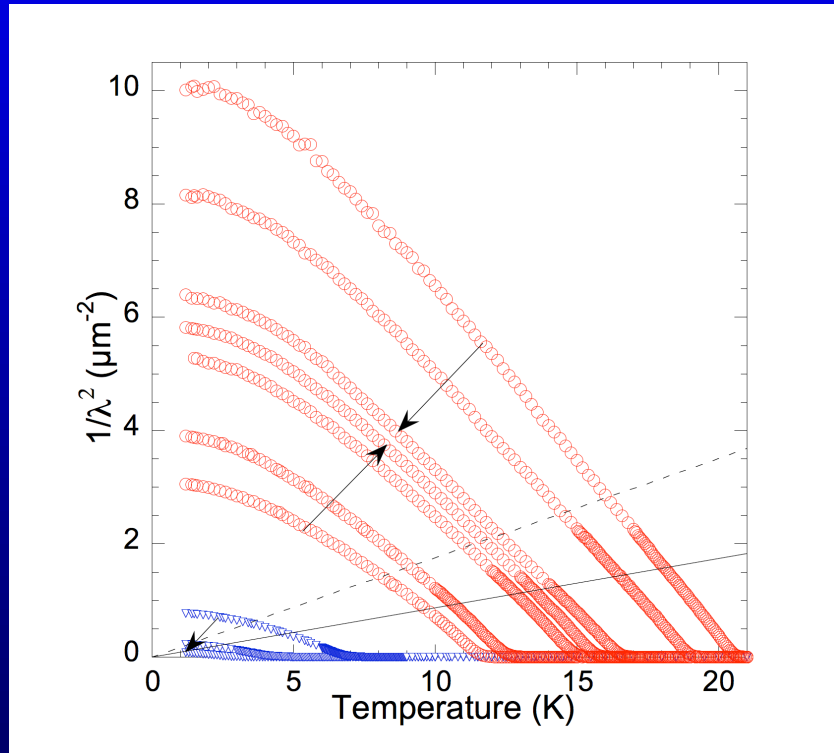
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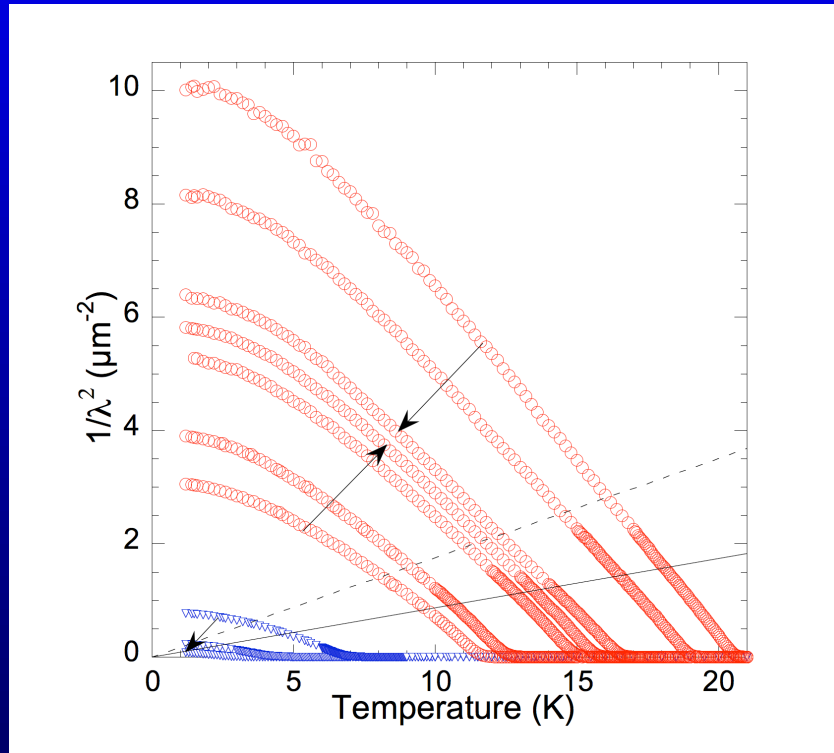


Data from UBC/SFU group [Broun *et al.* unpublished]

New *ab*-plane phenomenology



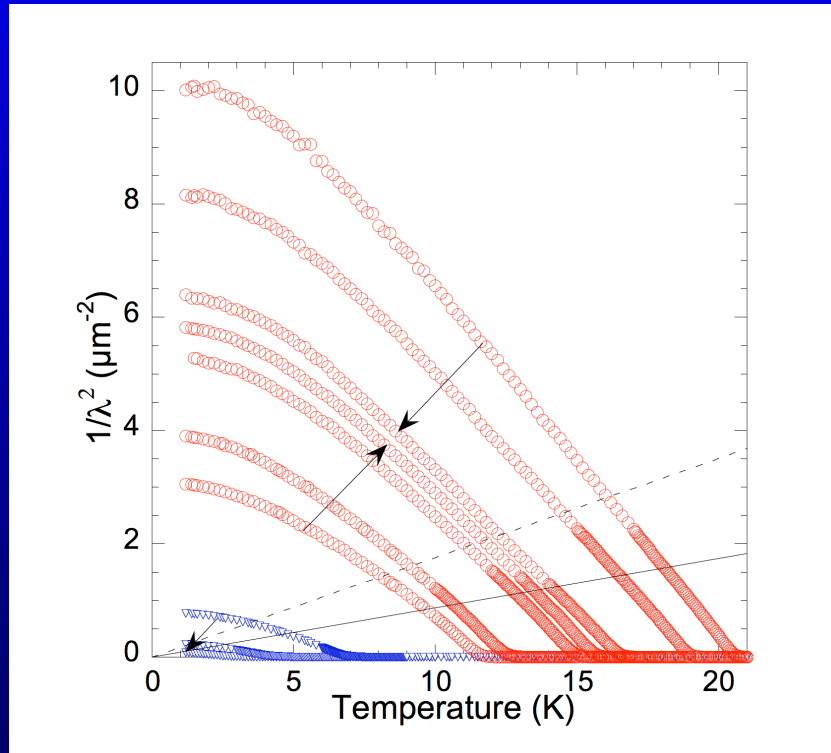
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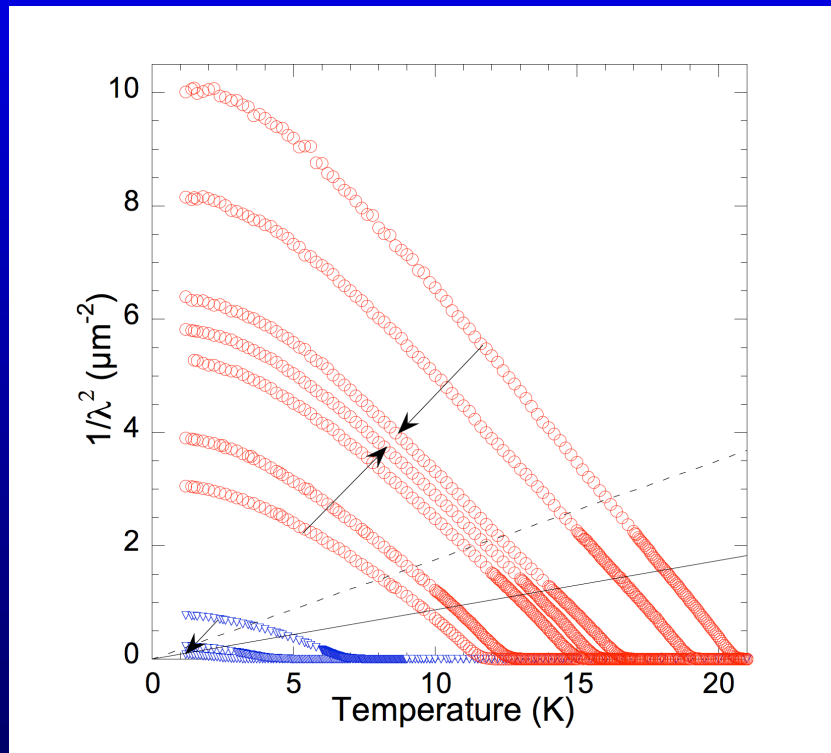
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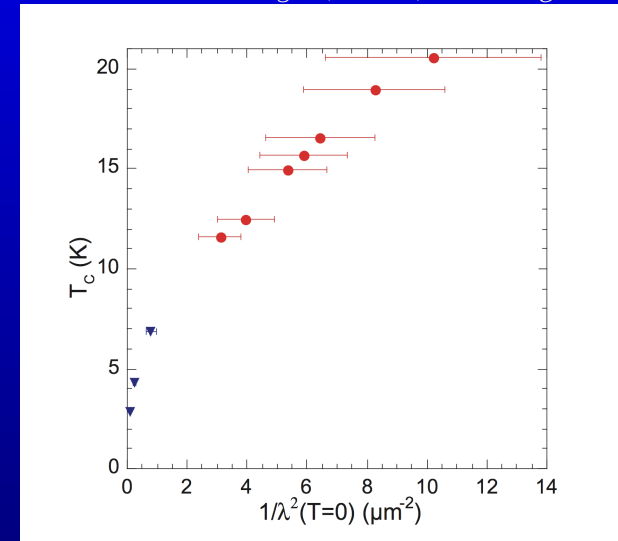
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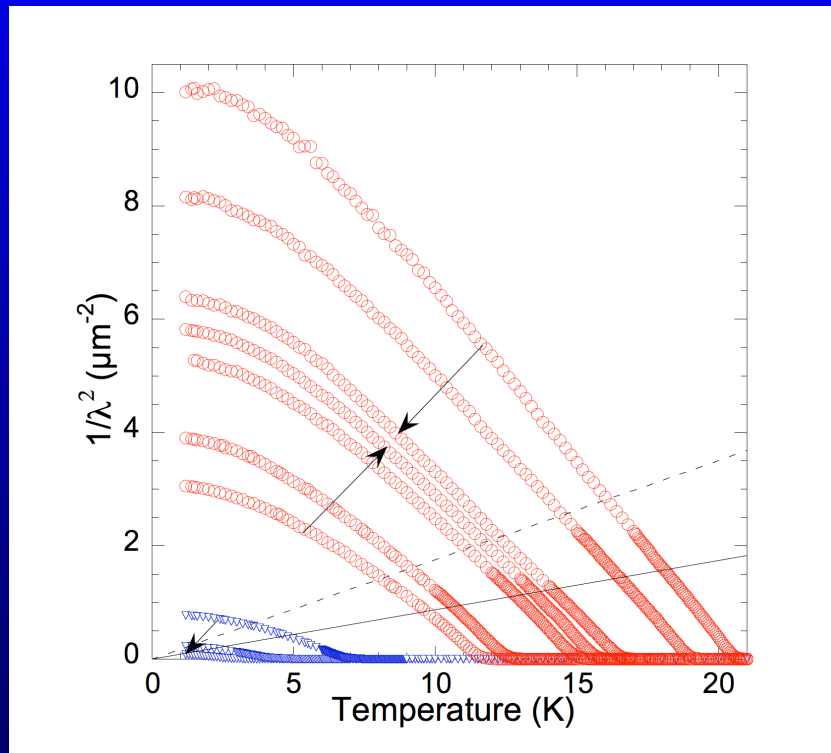
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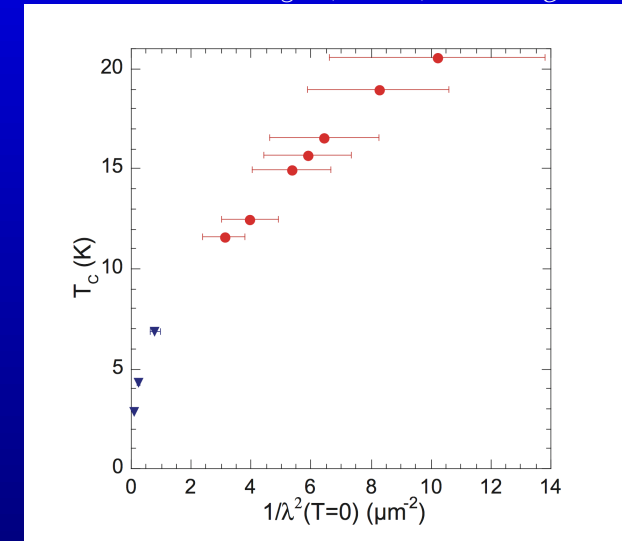
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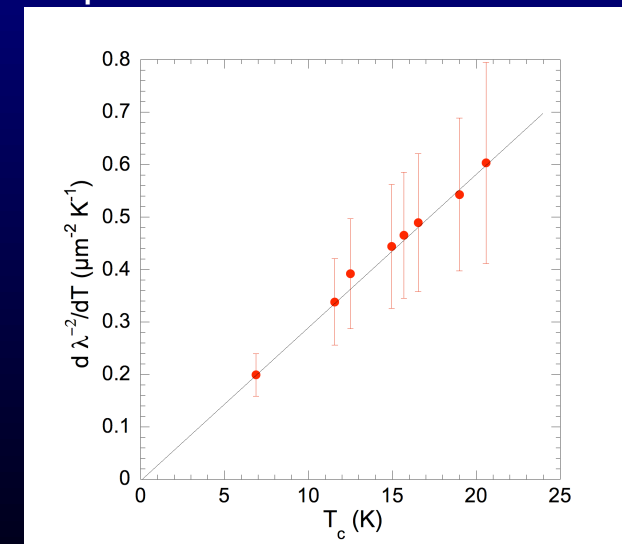
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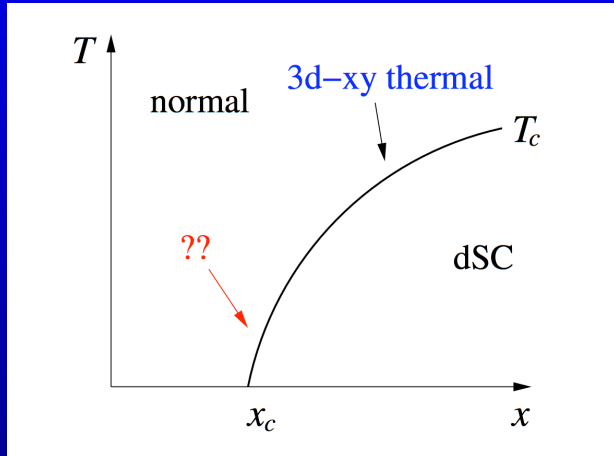
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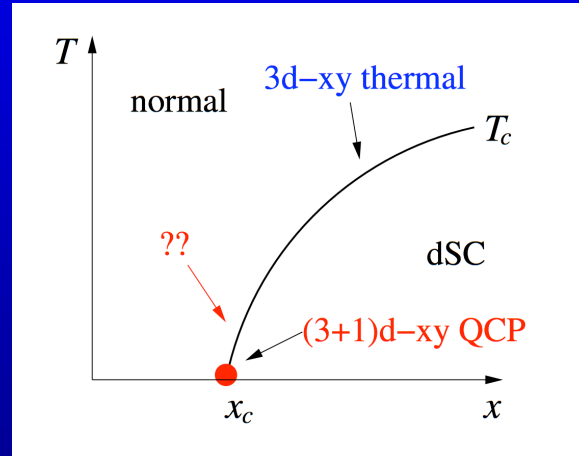
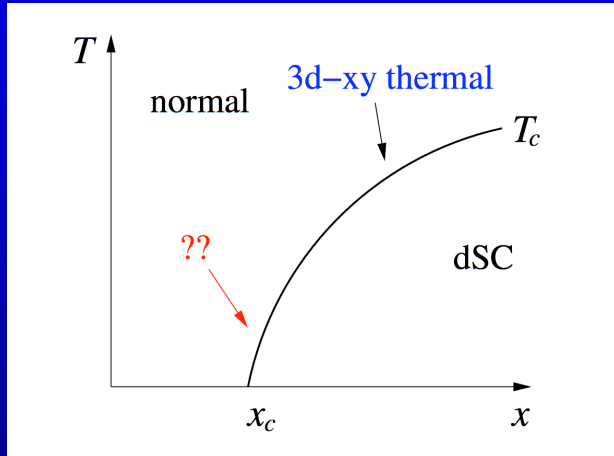
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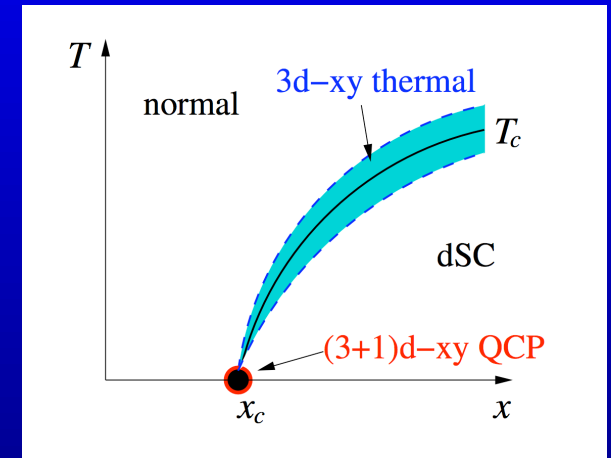
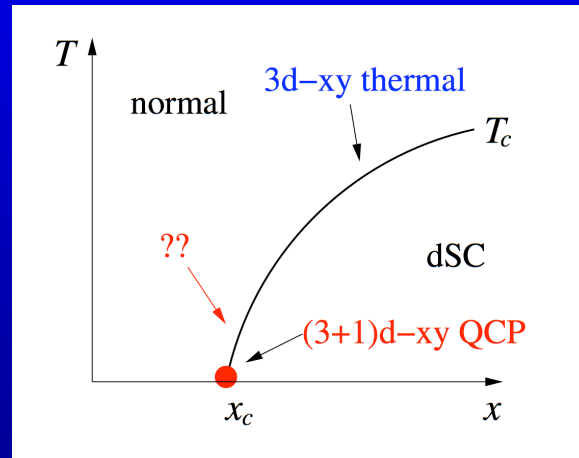
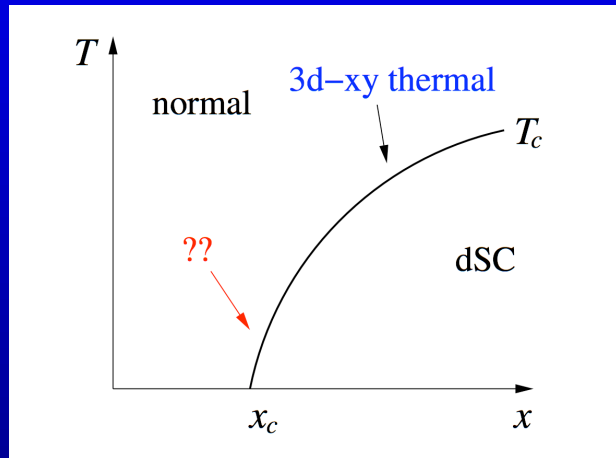
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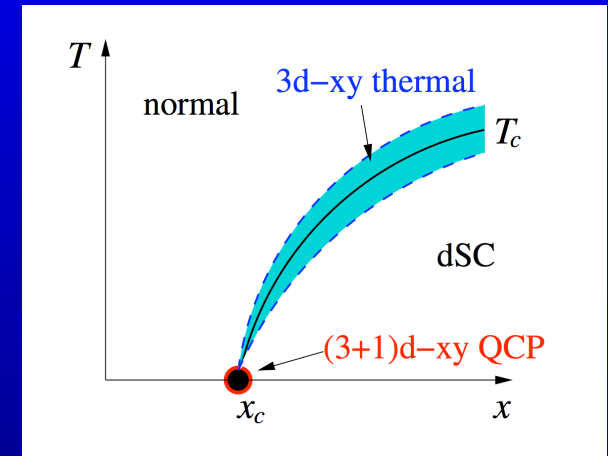
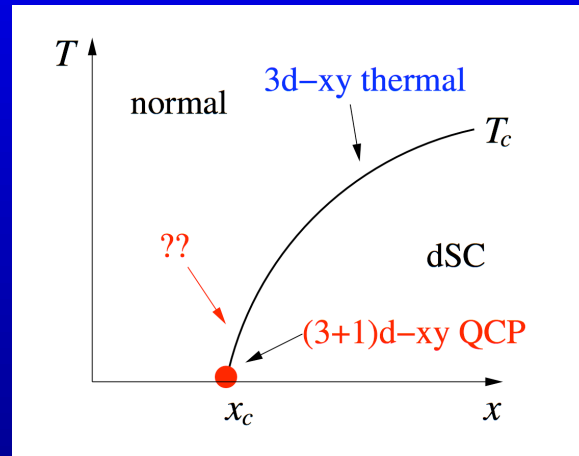
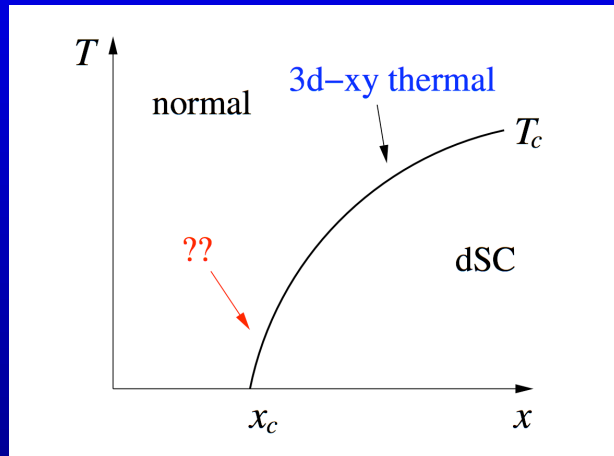


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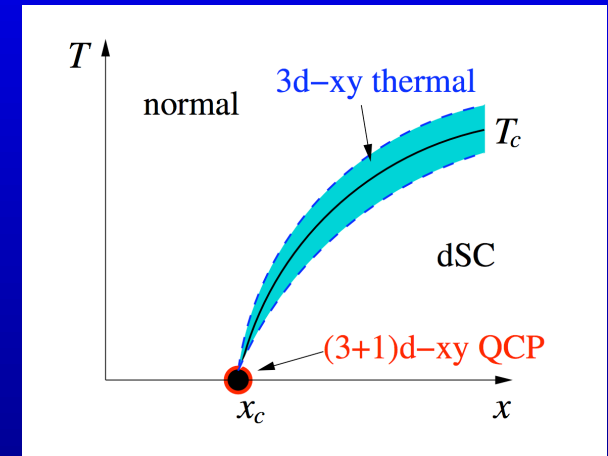
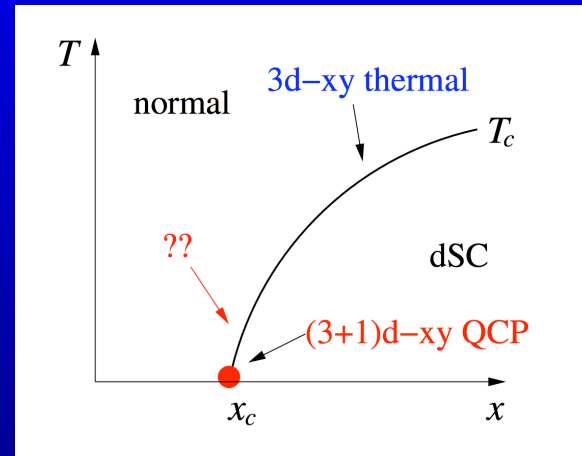
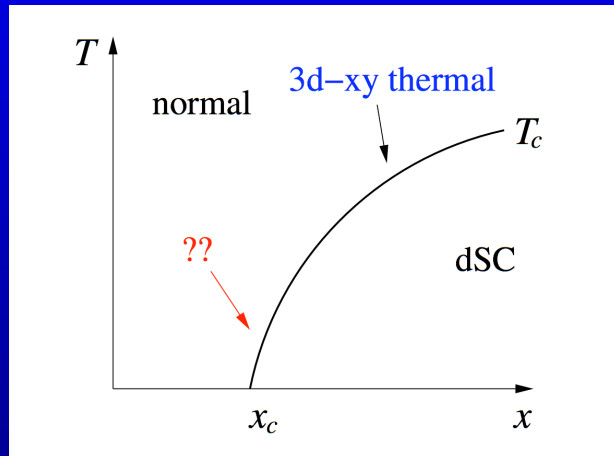
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- This QCP lives in **(3+1) dimensions**, "1" standing for the imaginary time τ .
- For xy-type models 4 is the upper critical dimension; one thus expects **mean field** critical behavior.

Consistency check

If the underdoped region is controlled by (3+1)D-XY quantum critical point then we should be able to understand the behavior of $\rho_s^{ab}(x, T)$ based on very general **scaling arguments**.

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For $1 \leq z \leq 2$ this is consistent with experiment!

Problems

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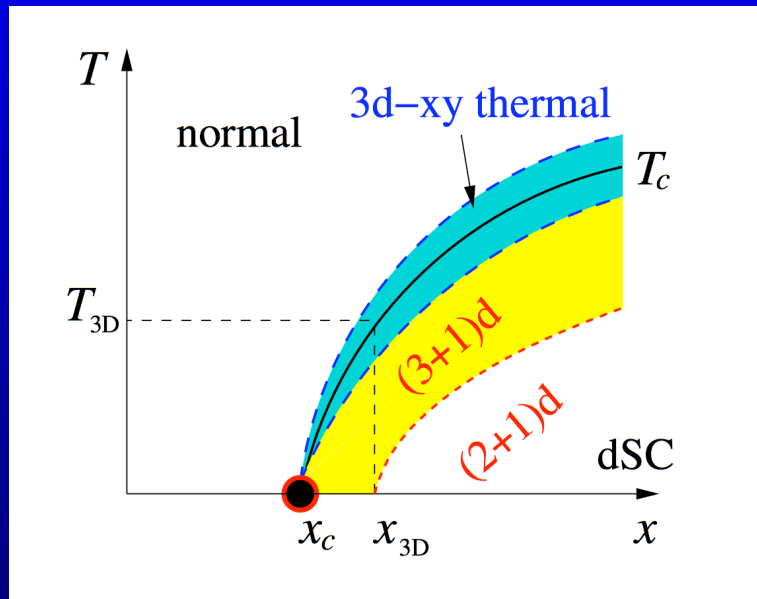
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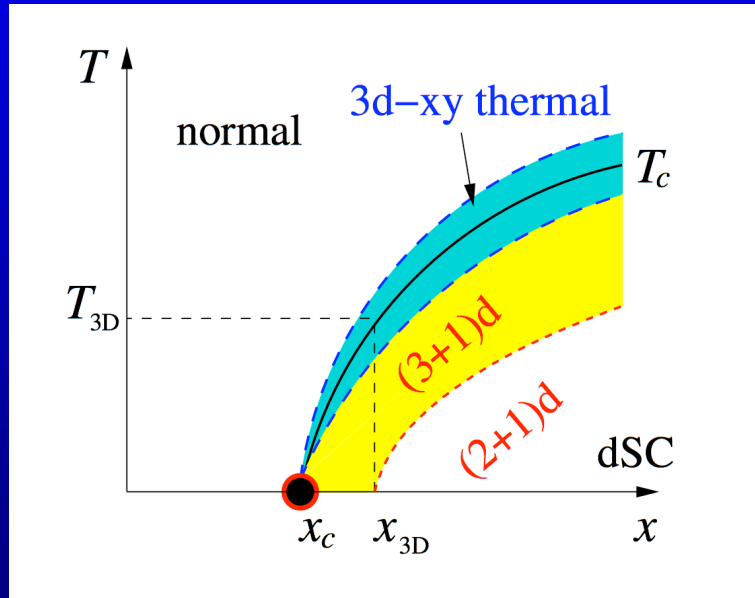
- Cuprates are strongly anisotropic; it is unclear how broad the (3+1)D critical region is.
- There is strong (linear) T -dependence of the “bare” superfluid density coming from quasiparticles which may invalidate the scaling laws.

2D - 3D crossover



At $T = 0$, away from QCP, fluctuations in CuO_2 layers are decoupled and system behaves 2 dimensionally.

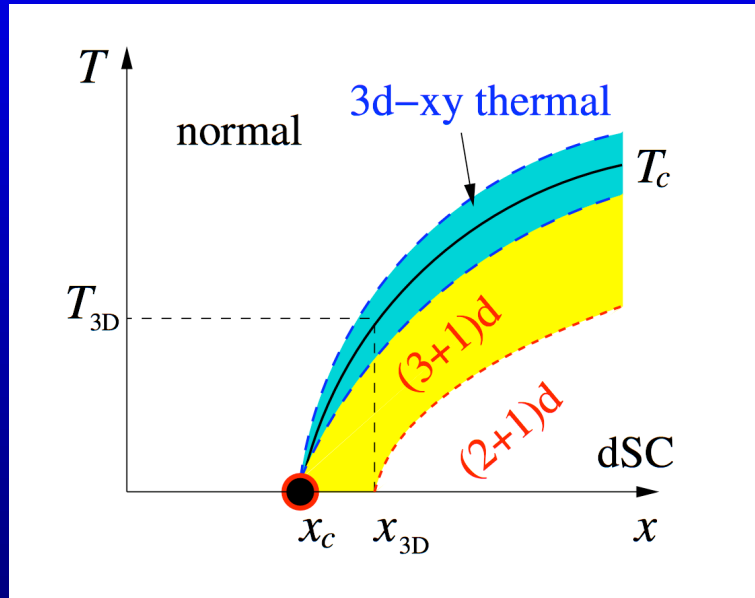
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For YBCO we estimate, using $\xi_c \approx \lambda_{ab}^2 / \lambda_c \kappa$,

$$T_{3D} \approx 5 - 10\text{K}.$$

Quantum XY model

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The simplest model showing XY-type critical behavior is given by the Hamiltonian

$$H_{\text{XY}} = \frac{1}{2} \sum_{ij} \hat{n}_i V_{ij} \hat{n}_j - \frac{1}{2} \sum_{ij} J_{ij} \cos(\hat{\varphi}_i - \hat{\varphi}_j).$$

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$$H_{\text{XY}} = \frac{1}{2} \sum_{ij} \hat{n}_i V_{ij} \hat{n}_j - \frac{1}{2} \sum_{ij} J_{ij} \cos(\hat{\varphi}_i - \hat{\varphi}_j).$$

Here \hat{n}_i and $\hat{\varphi}_i$ are the number and phase operators representing Cooper pairs on site \mathbf{r}_i of a cubic lattice and are quantum mechanically **conjugate variables**:

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Quantum XY model

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The sites \mathbf{r}_i do not necessarily represent individual Cu atoms; rather one should think in terms of “coarse grained” lattice model valid at long lengthscales where microscopic details no longer matter.

- The first term in H_{XY} describes **interactions** between Cooper pairs; we take

$$V_{ij} = U\delta_{ij} + (1 - \delta_{ij})\frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

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- In the absence of interactions J clearly must be identified as the **superfluid density**. We thus take

$$J = J_0 - \alpha T$$

with $\alpha = (2 \ln 2)v_F/v_\Delta$, as in the BCS d -wave superconductor.

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It also naturally yields the shrinking classical fluctuation region with decreasing T_c .

Self-consistent harmonic approximation

The idea is to replace the XY Hamiltonian

$$H_{\text{XY}} = \frac{1}{2} \sum_{ij} \hat{n}_i V_{ij} \hat{n}_j - J \sum_{\langle ij \rangle} \cos(\hat{\varphi}_i - \hat{\varphi}_j).$$

by the “trial” harmonic Hamiltonian

$$H_{\text{har}} = \frac{1}{2} \sum_{ij} \hat{n}_i V_{ij} \hat{n}_j + \frac{1}{2} K \sum_{\langle ij \rangle} (\hat{\varphi}_i - \hat{\varphi}_j)^2.$$

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This is just a variational principle which can be extended to $T > 0$ case using the Gibbs-Bogolyubov inequality $F \leq F_{\text{har}} + \langle H - H_{\text{har}} \rangle_{\text{har}}$.

H_{har} is **quadratic** in \hat{n}_i and $\hat{\varphi}_j$ and can thus be easily diagonalized:

$$H_{\text{har}} = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} \left(a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \frac{1}{2} \right)$$

with the frequencies

$$\hbar\omega_{\mathbf{q}} = 2\sqrt{KZ_{\mathbf{q}}V_{\mathbf{q}}}, \quad Z_{\mathbf{q}} = \sin^2(q_x/2) + \sin^2(q_y/2) + \sin^2(q_z/2).$$

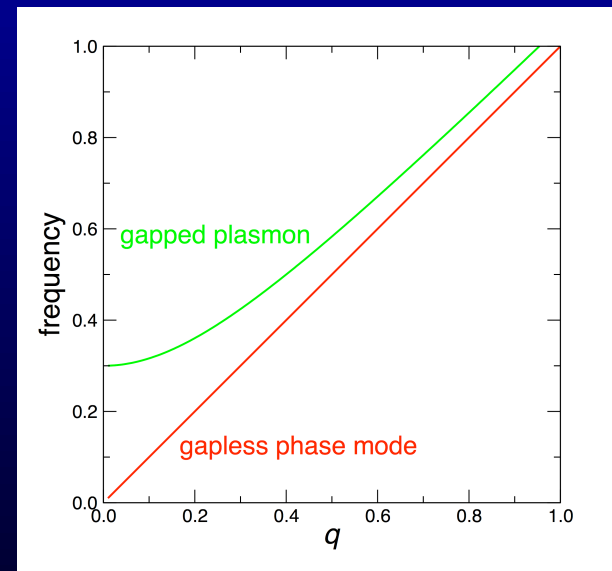
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- For short range interactions $V_{\mathbf{q}} \rightarrow \text{const}$ as $q \rightarrow 0$; we have $\omega_{\mathbf{q}} \sim q$, i.e. acoustic phase mode.
- For Coulomb interactions $V_{\mathbf{q}} \sim 1/q^2$ as $q \rightarrow 0$; we have $\omega_{\mathbf{q}} \rightarrow \omega_{\text{pl}}$, i.e. gapped plasma mode.



Simple power counting shows that at low T the contribution from the phase mode to the superfluid density is

$$\delta\rho_s^{\text{ph}} \sim \begin{cases} T^3, & \text{short range interaction} \\ e^{-\omega_{\text{pl}}/T}, & \text{Coulomb interaction} \end{cases}$$

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However, as we shall see, quantum fluctuations will strongly renormalize both the amplitude J_0 and the slope α .

SCHA: the results

Using the identity $\langle \cos(\hat{\varphi}_i - \hat{\varphi}_j) \rangle_{\text{har}} = \exp \left[-\frac{1}{2} \langle (\hat{\varphi}_i - \hat{\varphi}_j)^2 \rangle_{\text{har}} \right]$, valid for harmonic Hamiltonians, we obtain

$$E_{\text{har}} = \langle H_{XY} \rangle_{\text{har}} = \sqrt{KS} - J e^{-\sqrt{S/K}},$$

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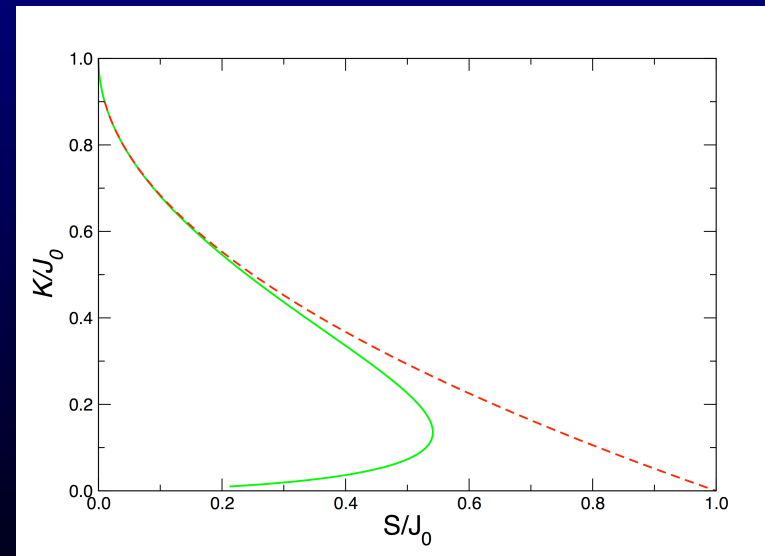
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Minimizing E_{har} with respect to K we obtain

$$K = J e^{-\sqrt{S/K}} \simeq J(1 - \sqrt{S/J}).$$



To obtain the leading temperature dependence substitute $J = J_0 - \alpha T$ and expand to leading order in T :

$$\rho_s(x, T) = K \simeq J_0 \left(1 - \sqrt{\frac{S}{J_0}} \right) - \alpha T \left(1 - \frac{1}{2} \sqrt{\frac{S}{J_0}} \right).$$

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In particular, for small $\sqrt{S/J_0}$ the above expression is consistent with experimentally observed behavior

$$\rho_s^{ab}(x, T) \simeq J_0 x^2 - \alpha x T,$$

if we identify $x \simeq \left(1 - \frac{1}{2} \sqrt{\frac{S}{J_0}}\right)$.

How about the region $S \approx J_0$?

In this regime one can construct a “critical” theory of strong phase fluctuations [Doniach, PRB 24, 5063 (1981)] as an expansion in small order parameter $\psi(x, \tau)$.

This leads to a quantum Ginzburg-Landau action

$$S = \int_0^\beta d\tau \int d^3x \left\{ r|\psi|^2 + \frac{1}{2}u|\psi|^4 + \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2c^2}|\partial_\tau\psi|^2 \right\},$$

with parameters r , u and c given as functions of J and S .

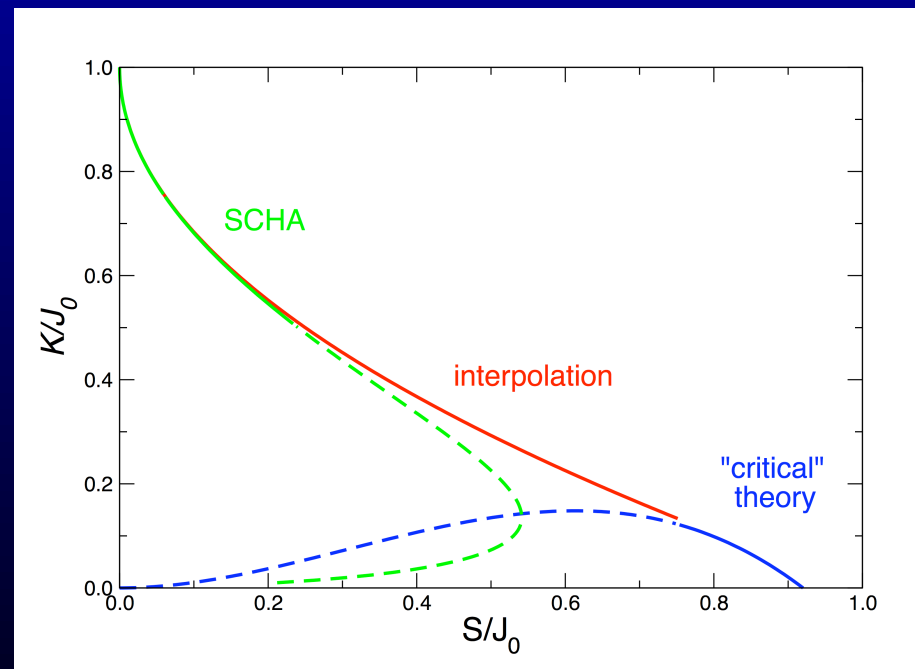
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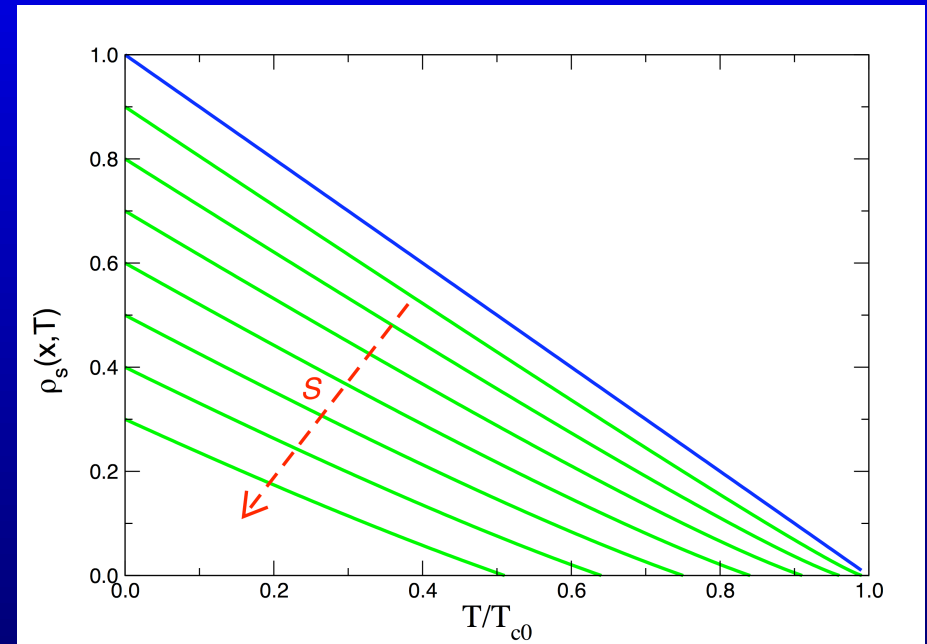
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Combining SCHA with this critical theory provides a consistent picture for the suppression of ρ_s by quantum fluctuations.



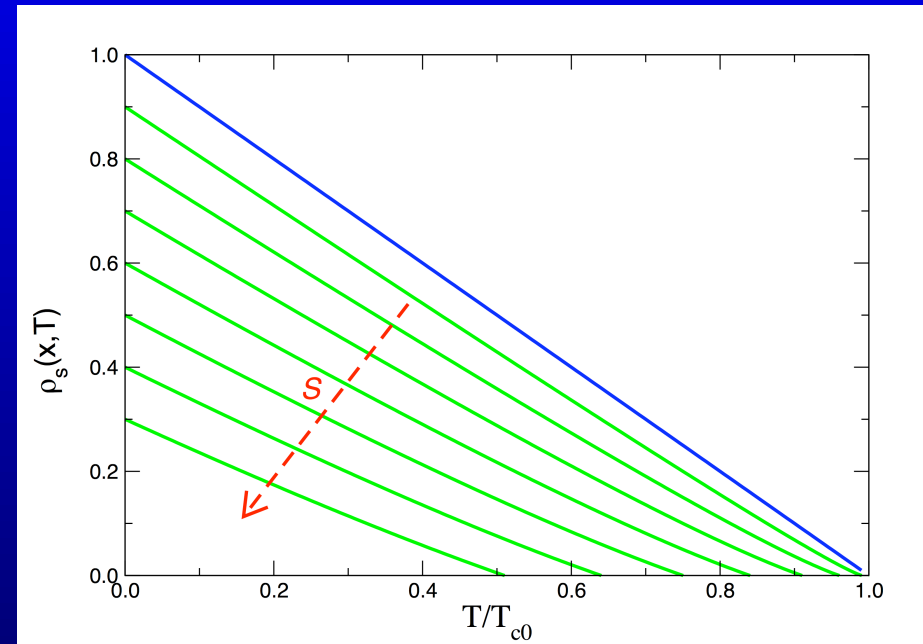
Summary

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We are currently investigating implications of this model for other physical **observables**, namely specific heat, *c*-axis superfluid density, fluctuation diamagnetism, thermal and electrical conductivity.

Conclusions

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- This phenomenology puts severe constraints on microscopic models of underdoped cuprates.