Homework 5

1. For an N-site chain of spins $\frac{1}{2}$, described by a FM Heisenberg Hamiltonian

$$\mathcal{H} = -J \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} - \frac{1}{4} \right]$$

show:

(a) that $|FM\rangle = |+\frac{1}{2}, ..., +\frac{1}{2}\rangle$ (FM state with all spins up) is a true eigenstate. Can you show that this must be the ground-state?

(b) that the one-magnon state

$$|q\rangle = \frac{1}{\sqrt{N}} \sum_q e^{iqR_i} S_i^- |FM\rangle$$

is also a true eigenstate. Find the magnon energy $\omega(q)$ (set $\hbar = 1$). You might want to see how the problem generalizes for spin S.

2. Let b, b^+ be bosonic operators, $[b, b^+] = 1$. (i) Find the coherent state $|\alpha\rangle$ which is an eigenstate of the annihilation operator, $b|\alpha\rangle = \alpha|\alpha\rangle$; (ii) Find the probability $P_n = |\langle n|\alpha\rangle|^2$ to have n bosons in this state. (iii) find the average number of bosons in this state.

3. For the Hamiltonian:

$$\mathcal{H} = \Omega b^+ b + g c^+ c \left(b^+ + b \right)$$

calculate the Green's function

$$G(\omega) = \langle 0 | c \hat{G}(\omega) c^+ | 0 \rangle$$

where $|0\rangle$ is the vacuum (for both the bosons b and the fermions c) and $\hat{G}(\omega) = 1/(\omega - \mathcal{H} + i\eta)$. Find the associated spectral weight $A(\omega) = -\frac{1}{\pi} Im G(\omega)$ and sketch it.

4. Consider the spin-polaron model discussed in class:

$$\mathcal{H} = -J \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} - \frac{1}{4} \right] - t \sum_{i,\sigma} [c_{i,\sigma}^+ c_{i+1,\sigma} + h.c.] + J_0 \sum_i \vec{s}_i \cdot \vec{S}_i$$

where $\vec{S}_i$ describe spins at each site i of a 1D chain, and $\vec{s}_i = \frac{1}{2} \sum_{\alpha,\beta} c_{i,\alpha}^+ \vec{\sigma}_{\alpha\beta} c_{i,\beta}$ is the spin created by fermions at site i, and where J, t, J_0 are all positive quantities.

If a spin-down fermion is introduced in the lattice, find the spin-polaron dispersion in the strong-coupling limit (fill in the details for the calculation sketched in the notes).

5. Consider a generalization of the problem 4: we now have two interlaced lattices, one (labeled by integers i) which hosts the impurity spins and one (labeled by half-integers $i + \frac{1}{2}$) which hosts the hopping electron:

$$\mathcal{H} = -J \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} - \frac{1}{4} \right] - t \sum_{i,\sigma} [c_{i+\frac{1}{2},\sigma}^+ c_{i+\frac{3}{2},\sigma} + h.c.] + J_0 \sum_i \vec{s}_{i+\frac{1}{2}} \cdot \left[\vec{S}_i + \vec{S}_{i+1} \right]$$

– in this case, the spin of the fermion couples equally strongly to both its neighboring spins. Again, we add a single spin-down electron to the FM chain. In the strong coupling limit $J_0 \gg t, J$ find (a) the structure of the polaron wavefunction with momentum k; and (b) the effective mass of this polaron.