

Brief review of Newtonian formalism

The following is a brief review of material that is hopefully already well known. This is only to understand some of the difficulties of this approach, and contrast with the new Lagrangian and Hamiltonian formulations that we will learn in this course. We will not use the Newtonian formulation of mechanics in this course.

1 Newton's Laws of Motion

First law: If the net force acting on a body is zero, then the body will remain in uniform motion (or at rest).

This law actually holds only in **inertial reference frames IRF**. As such, it can be also used as a definition of an IRF; in a IRF, a body experiencing a zero net force (also known as a free body) will remain at rest or in uniform motion.

It follows that the definition of the IRF is not unique: in any other reference frame which moves with a constant speed with respect to an inertial reference frame, the free body will also remain in uniform motion (or rest). As such, there are an infinite number of IRFs moving uniformly with respect to one another.

Galileo's principle states that in all these IRFs the properties of space and time are the same, and therefore the laws of mechanics are the same (as described below).

One thing that seems to not be all that clear to some of you is that the laws stated below DO NOT HOLD in non-inertial reference frames. I will focus on this aspect in class. In the remainder of this review, we will only consider inertial reference frames.

Second law: The rate of change of momentum equals the net force acting on the body.

$$\frac{d\vec{p}}{dt} = \vec{F} \quad (1)$$

The momentum is defined as $\vec{p} = m\vec{v}$, where \vec{v} is the speed of the body. m is called mass and is a measure of the inertia of the body. In classical mechanics we assume the mass of a body to be independent of its speed, and therefore we can rewrite Eq. (1) as:

$$m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

Third law: Every action has a reaction of equal magnitude, but acting in the opposite direction.

2 Solving problems within Newtonian formalism

The second law provides the basis for finding the dynamics of a body (i.e. how its position and speed vary in time) if we know the total net force acting upon it.

Most generally, a force may depend on the position and speed of an object, as well as time $\vec{F}(\vec{r}, \vec{v}, t)$. As a result, the second law can be rewritten as three coupled, second-order differential

equations of the unknown function $\vec{r}(t)$.

$$m \frac{d^2x}{dt^2} = F_x \left(\vec{r}, \frac{d\vec{r}}{dt}, t \right); \quad m \frac{d^2y}{dt^2} = F_y \left(\vec{r}, \frac{d\vec{r}}{dt}, t \right); \quad m \frac{d^2z}{dt^2} = F_z \left(\vec{r}, \frac{d\vec{r}}{dt}, t \right)$$

A unique solution can be found if initial conditions at a time $t = t_0$, namely $\vec{r}(t_0) = \vec{r}_0$, $\vec{v}(t_0) = \vec{v}_0$, are specified.

The net force \vec{F} acting on the body is the sum of all the forces acting on it; the origin of these forces are interactions with other bodies. The types of forces encountered in classical mechanics are:

1. Gravitational force: the gravitational attraction force experienced by a body of mass m found at a position \vec{r} with respect to another body of mass M is (see Fig. 1).

$$\vec{F}_g = -\frac{GmM}{r^3}\vec{r}; \quad |\vec{F}_g| = \frac{GmM}{r^2}$$

From the 3rd principle, it follows that the body of mass M is acted upon by a force $-\vec{F}$.

If the body of mass m is very close to the surface of the (large) body of mass M (planet Earth, for instance), then the gravitational force is roughly independent of the position since $r \sim R$, the radius of the Earth (see Fig. 2), and

$$\vec{F}_g = -mg\vec{e}_z$$

Here, $g = GM_E/R^2$ is the gravitational acceleration.

2. Elastic force: if the body is linked to a spring with a spring constant k , then if the spring is elongated (compressed) by a distance $\delta\vec{r}$ from its equilibrium length, the elastic force which acts on the body is:

$$\vec{F}_{el} = -k\delta\vec{r}$$

i.e. it acts in the direction opposite to the distortion.

3. Friction forces: such forces may appear when two bodies are in contact, or when a body moves through a viscous medium. Strictly speaking, in these conditions the dynamics of the system is no longer a purely mechanical problem; some of the energy of the moving body is converted to heat and that leads us into thermodynamics!

The friction force between two bodies can be **static** or **dynamic**, depending on whether the objects move with respect to one another. The **static friction force** is oriented along the contact surface in such a way as to oppose the tendency of motion of the bodies. Its magnitude satisfies the inequality $F_s \leq \mu_s N$, where μ_s is the coefficient of static friction, and N is the magnitude of the normal force between the two bodies. The **dynamic friction force** is oriented along the contact surface, opposite to the direction of motion of the body it acts upon. Its magnitude is given by $F_f = \mu_k N$, where $\mu_k < \mu_s$ is the coefficient of (dynamic) friction.

The **drag** forces acting on a body moving through a viscous medium are rather complicated. A simple expression is obtained for a sphere of radius r moving slowly at speed v (such that no turbulence is generated). In this case, $\vec{F}_d = -6\pi\eta r\vec{v}$, where η is the coefficient of viscosity. An

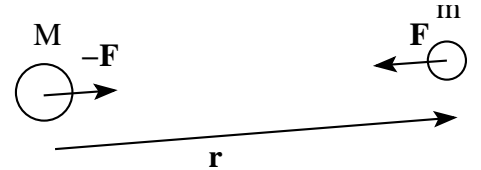


Fig 1. Gravitational force.

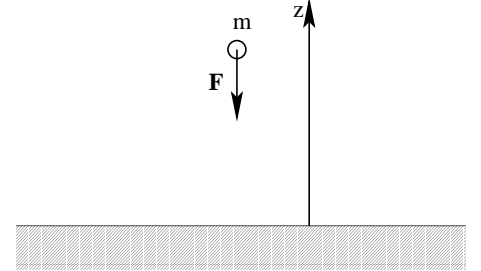


Fig 2. Gravitational force near the surface of a large object.

excellent elementary discussion of what happens for fast moving bodies is given in the Feynman lectures (Volume II sect 41).

4. Normal (constraint) forces: such forces act on a body which is constrained to move in a certain way by other bodies. Typical examples are:

a) if the body is tied to a fully extended string or wire, there is a force, called **tension**, acting upon the body, along the direction of the wire. The magnitude of the force is such as to prevent the motion of the body outside the region imposed by the constraint (length of wire).

b) if the body is in contact with a firm surface, there is a **normal force** acting along the perpendicular to the surface of contact. Its magnitude is such as to prevent the motion of one body through the other one.

The general characteristic of constraint forces is that their magnitude is not apriorily known (we have to compute it), but we know precisely their effect on the dynamics. This is to be contrasted with the gravitational, elastic and dynamic friction forces mentioned above, whose magnitude is known from the outset, however their effect on the dynamics is not (that is usually what we are trying to figure out, what happens to the body when these forces act on it).

Newtonian mechanics becomes complicated to use for systems with constraints, especially if the system considered contains several interacting objects. The prescription is to write the second law [Eq. (1)] for each body in the system. Thus, for a total of N bodies, there is a total of $3N$ second-order differential equations. These equations depend on normal (constraint) forces which are not known at the outset of the calculation. Thus, part of the $3N$ equations must be solved to find the values of these constraint forces, and then the remaining equations actually give us the dynamics of the system along the allowed directions.

Since we are generally not interested in the values of the normal forces, it would be very advantageous if we only needed to solve a smaller number of equations, describing the pure dynamics of the system.

Example: consider the equation of motion of a body of mass m tied to a string of length l . We assume in-plane motion, and choose an inertial system of coordinates as shown in Fig. 3. There are two forces acting on the body: gravitational force (known) and tension T (unknown).

In the Newtonian formalism, we have to write ($z = 0$):

$$m \frac{d^2x}{dt^2} = mg - T \cos \phi, \quad m \frac{d^2y}{dt^2} = -T \sin \phi$$

In this reference frame, the constrain condition is (i.e. T must be precisely the value for which) $x^2 + y^2 = l^2$, or

$$x = l \cos \phi, \quad y = l \sin \phi$$

and the equations become:

$$\begin{aligned} -ml \cos \phi \dot{\phi}^2 - ml \sin \phi \ddot{\phi} &= mg - T \cos \phi \\ -ml \sin \phi \dot{\phi}^2 + ml \cos \phi \ddot{\phi} &= -T \sin \phi \end{aligned}$$

We should now solve for T from one equation, and then plug the answer in the second one, to

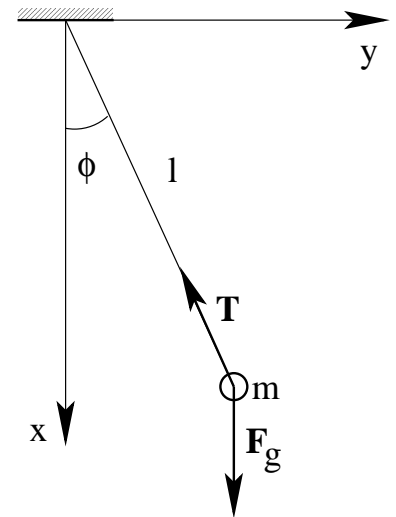


Fig 3. Body of mass m tied to a string.

find an equation describing how ϕ varies in time. This procedure is rather ugly, so we will use a trick to eliminate the unknown T by multiplying first equation by $\sin \phi$ and the second by $\cos \phi$, and subtracting them. The final result is

$$m l \ddot{\phi} = -mg \sin \phi$$

which must be integrated to find the answer. We knew from the beginning that because of the constraint, only the angle ϕ can vary and therefore the dynamics of the system must be described by a single differential equation for ϕ . As long as we're interested in finding only the dynamics, but not the constraint force T , it is a waste of time to start with two equations with an extra unknown (the constraint force T) and then reduce them to a single equation. The problem becomes serious when there are many bodies and many constraints; each constraint will introduce a new unknown constraint force, and we must solve for all of them before we can eliminate them and find the equations describing only the dynamics of the system.

Surely there must be a way to avoid this extra work! Indeed, one of the convenient features of the Lagrangian and Hamiltonian formulations of mechanics is that they lead directly to the minimum number of equations necessary to describe the pure dynamics of the system. However, there is an even more important advantage of these formulations as compared to Newtonian mechanics: while the Newtonian mechanics is based on the notion of force, the Lagrangian/Hamiltonian formulations are based on the Lagrangian/Hamiltonian of the system, which is a quantity with energy-like character (we will learn the precise definitions of these quantities in the coming lectures). As such, the Lagrangian/Hamiltonian approach can be easily generalized to describe other systems, such as continuous (deformable) media, electromagnetic fields and general classical field theories etc, where the notion of force does not exist. More importantly, these approaches can be generalized to treat quantum mechanics, quantum field theories (such as quantum electrodynamics) etc. In fact, you've probably already heard of "Hamiltonians" in your intro quantum mechanics course!

However, as far as classical mechanics goes, Newtonian, Lagrangian and Hamiltonian formalisms are perfectly equivalent. This is why classical mechanics is the perfect discipline to learn about Lagrangian/Hamiltonian formalisms, since we have something (the Newtonian formalism) to compare our results against and validate them.

3 Conservative forces and potential energy

A force $\vec{F}(\vec{r})$ is **conservative** if a **potential** $U(\vec{r})$ exists, such that $\vec{F}(\vec{r}) = -\nabla \cdot U(\vec{r})$. Here $\nabla = \vec{e}_x \partial/\partial x + \vec{e}_y \partial/\partial y + \vec{e}_z \partial/\partial z$ and therefore $\nabla \cdot U(\vec{r}) = \vec{e}_x \partial U/\partial x + \vec{e}_y \partial U/\partial y + \vec{e}_z \partial U/\partial z$. Potential energy is defined only up to a constant (in fact, up to a $\nabla \times f$ contribution).

If a point-like object is acted upon only by conservative forces, one can show that its total energy $mv^2/2 + U(\vec{r})$ is conserved (i.e., it is constant in time).

Types of potential energies:

- (1) elastic potential energy (of a spring)

$$U = \frac{k(\delta\vec{r})^2}{2}$$

It is positive (energy is stored in a distorted spring, whether it is compressed or extended), and it increases like the square of the distortion.

(2) gravitational potential energy:

$$U(\vec{r}) = -\frac{GmM}{r} \text{ respectively } U(z) = mgz$$

(the coordinates are defined in Figs. 1 and 2). Since the two masses attract, the further away they are from one another (larger r or z) the larger is their potential energy. To remember the proper sign, you need to keep in mind that the more stable configurations have lower potential energies.

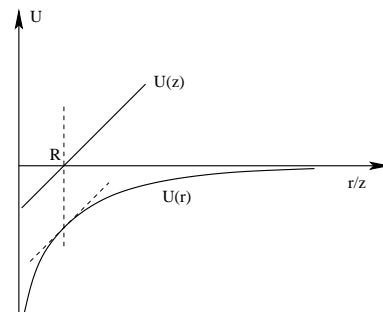


Fig 4. Gravitational potential energy.

4 Rotating bodies: angular momentum, moments of inertia, etc.

Let me re-iterate that the only pieces of information from this review, that we will actually use during the course, are Galileo's principle and the two types of potential energies just listed. All the rest is just a reminder of Newtonian mechanics, for your convenience.

Later on we will study rotating rigid bodies, again using Lagrangian/Hamiltonian formalism. For completeness, let me briefly summarize it for you here the Newtonian version of how to deal with such problems. While we will do things differently, some of the basic concepts, such as angular momentum and moment of inertia, will certainly show up in our course.

With respect to an IRF, we know that if we have an object of mass m in linear motion, its dynamics is characterized by $\vec{F} = \frac{d\vec{p}}{dt}$, where \vec{F} is the net force acting on it, and $\vec{p} = m\vec{v}$ is its momentum. Of course, \vec{v} is the speed, which measures how the position changes in time.

For rotating bodies, we cannot talk about "a speed" of the object – unlike in translational motion where all parts of the object move at the same speed, in rotational motion different parts have different speeds. For instance, consider the rod shown \rightarrow , for which points on the axis of motion are at rest, while parts which are further apart from it move faster (I assume the object to be rigid).

Because of this, we use angular velocity $\vec{\omega} = \vec{n} \frac{d\phi}{dt}$ which is the same for the entire body. Its magnitude is the rate $\frac{d\phi}{dt}$ at which the angle changes, and the direction \vec{n} shows the axis of rotation.

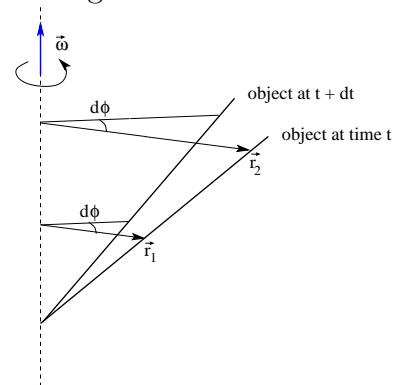


Fig 5. Object (simple rod) rotated about an axis (dashed line).

Instead of momentum $\vec{p} = m\vec{v}$ we now use the angular momentum $\vec{L} = I\vec{\omega}$. I is called the moment of inertia, and depends not only on the total mass of the body, but also on its shape and how is the mass distributed in it. To calculate it, we split the object into very small parts of mass δm_i . Then: $I = \sum_i \delta m_i r_i^2$ where r_i is the distance between the piece i and the axis of rotation. I've drawn these distances for two such pieces.

Finally, let δf_i be the force acting on piece i . Clearly, not all forces have the same effect on the rotation: the ones that act further away from the axis and in a tangential direction will be more effective in increasing the rate of rotation. On the other hand, forces acting in a radial direction play no role in the rotation. This is why, instead of the net force $\vec{F} = \sum_i \delta \vec{f}_i$, the relevant quantity for

rotation is the torque $\vec{\tau} = \sum_i \vec{r}_i \times \delta \vec{f}_i$. In terms of this, Newton's second law becomes:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

This correspondence between translational and rotational motion:

$$\begin{aligned}\vec{v} &\rightarrow \vec{\omega}; \\ m &\rightarrow I; \\ \vec{p} = m\vec{v} &\rightarrow \vec{L} = I\vec{\omega}; \\ \vec{F} &\rightarrow \vec{\tau}; \\ \vec{F} = \frac{d\vec{p}}{dt} &\rightarrow \vec{\tau} = \frac{d\vec{L}}{dt}\end{aligned}$$

goes further. For example, the translational kinetic energy $\frac{\vec{p}^2}{2m} = \frac{m\vec{v}^2}{2}$ is replaced by rotational kinetic energy $\frac{\vec{L}^2}{2I} = \frac{I\vec{\omega}^2}{2}$, etc.

The big complication with rotational motion is what happens if the axis of rotation itself is moving in time (so far, I assumed it to be fixed). Then things become more interesting, because I is no longer a constant (remember that it depends on how far various pieces are from the axis. This is a constant quantity if the axis doesn't change, but not if the axis keeps changing). This is very different from the mass, which is always the same no matter the direction of motion. We will see in this course how we deal with such complications in an elegant way.