Special Relativity as a symmetry of nature

Einstein's Principle of Relativity states that the laws of physics are the same in all frames of reference. To understand the implications of this for quantum field theories, it is crucial to realize that this statement implies that there is a symmetry of nature above and beyond the usual translations, time translations, and rotations. Specifically, if we have any physical scenario (which satisfies the equations of motion/ the Schrodinger equation) and we consider another scenario equivalent to how the first scenario would appear to an observer moving at constant velocity, then the new scenario must also satisfy the same equations of motion / Schrodinger equation.

The transformation from one physical configuration to another one equivalent to how the first would appear in another frame of reference is known as a BOOST. For example, applying a boost to a state with a particle at rest gives us a state with a particle moving at some constant velocity. The set of all possible boosts, together with all possible rotations is known as the set of LORENTZ TRANSFORMATIONS. Mathematically, these transformations form a group, known as the LORENTZ GROUP. The set of all possible boosts, rotations and translations form the POINCARÉ TRANSFORMATIONS, which are the elements of the POINCARÉ GROUP.

Explicit form of the Poincaré transformations

To describe explicitly how the Poincare transformations affect coordinates and fields, it is useful to collect the time and space coordinates into a 4-VECTOR

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} ,$$

where μ is taken to run from 0 (for the time coordinate) to 3. With this notation, we can write the action of a general Poincare transformation as

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1}$$

where a^μ parameterize the translations and time translations, and $\Lambda^\mu{}_\nu$ is a matrix describing the rotations and boosts. For example, a rotation around the z axis corresponds to

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

while a boost with velocity v in the x direction corresponds to

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Here, we define $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$.

Transformation of physical quantities under Lorentz tranformations

Physical quantities all have some particular transformation rule under Lorentz transformations. For certain quantities, such as proper time and total charge, there is no change when we do a Lorentz transformation. Other quantities, such as energy and momentum transform in the same way as the coordinates once we combine them into a 4-vector

$$p^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} .$$

For constructing quantum field theories, we will be mostly interested in the transformation properties of fields. Just as for the more restricted case of rotations (see class notes), we can have SCALAR FIELDS that transform as

$$\tilde{\phi}(\Lambda x) = \phi(x) ,$$

defined so that the new field at a transformed coordinate location is equal to the old field at the old location. We can also have 4-VECTOR fields whose components are transformed like the coordinates (in a similar way to the rotational vector fields)

$$\tilde{\phi}^\mu(\Lambda x) = \Lambda^\mu{}_\nu \phi^\nu(x) .$$

Eventually, we'll want to understand all the possibilities for Lorentz transformation rules on fields.

General Lorentz Transformations

The most general Lorentz transformation can be obtained by a combination of boosts and rotations in the various directions. It would be very ugly to write a matrix for the most general such transformation, but there is a very nice way to characterize the Lorentz transformations in general. First recall that rotations may be defined as the group of transformations $\vec{x} \to R\vec{x}$ that preserve distances $\Delta\vec{x} \cdot \Delta\vec{x}$ (so that distances are preserved). In special relativity, you learned that distances between events could be different in different frames of reference. Thus, general Lorentz transformations do not preserve dot products, but they do preserve the special combination known as the INVARIANT INTERVAL

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta\vec{x})^2 = \Delta x^\mu \eta_{\mu\nu} \Delta x^\nu$$

where we have defined

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

In fact, the Lorentz transformations can be defined as the set of transformations which preserve $(\Delta s)^2$, in the same way that the rotations can be defined as the set of transformations $\vec{x} \to R\vec{x}$ that preserve distances.[1] It is simple to show that a transformation (1) will preserve $(\Delta s)^2$ if and only if

$$\Lambda^{\alpha}{}_{\mu}\eta_{\alpha\beta}\Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu} \ ,$$

or in matrix language

$$\Lambda^T \eta \Lambda = \eta \ . \tag{2}$$

Notation

In expressions involving 4-vectors, we frequently have the matrix $\eta_{\mu\nu}$ multiplying a vector (or sandwiched between two vectors as in the invariant interval). To make equations simpler, it is convenient to use the notation

$$x_\mu = \eta_{\mu\nu} x^\nu \ .$$

Then, for example, we can write $\Delta s^2 = \Delta x_\mu \Delta x^\mu$.

By it's definition, x_μ is the same as x with minus signs on its spatial components. This difference means that the transformation rule for x_μ is slightly different:

$$x_\mu \to \Lambda_\mu{}^\nu x_\nu$$

where we have defined $\Lambda_\mu{}^\nu = \eta_{\mu\alpha}\Lambda^\alpha{}_\beta \eta^{\beta\nu}$ (here $\eta^{\mu\nu}$ is the same matrix as $\eta_{\mu\nu}$).

This notation is also used often in writing actions for fields. For example

$$\partial_\mu\phi\partial^\mu\phi \equiv \partial_\mu\phi\eta^{\mu\nu}\partial_\nu\phi \equiv \frac{\partial\phi}{\partial x^\mu}\eta^{\mu\nu}\frac{\partial\phi}{\partial x^\nu} = \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \vec{\nabla}\phi\cdot\vec{\nabla}\phi$$

It turns out that the components of the derivative of a field $\partial_\mu\phi = \frac{\partial\phi}{\partial x^\mu}$ transform in the same way as x_μ, so the derivative is normally written with a lower index.

Infinitesimal Lorentz Transformations

It is very useful to understand what the infinitesimal Lorentz transformations look like. Suppose that we have a family of Lorentz transformations $\Lambda(a)$, where $\Lambda(0)$ is the identity matrix (which we call 1). Then we must have

$$\Lambda(\epsilon) = 1 + \epsilon\omega + \mathcal{O}(\epsilon^2)$$

for some matrix ω. Plugging this into (2) we find that ω must satisfy

$$\omega^T\eta = -\eta\omega$$

[1] By this definition, Lorentz transformations include not only rotations and boosts, but two special transformations known as PARITY $(t, x, y, z) \to (t, -x, -y, -z)$ and TIME REVERSAL $(t, x, y, z) \to (-t, x, y, z)$.

or

$$-\eta\omega^T\eta = \omega \ .$$

It is easy to show that the general solution of this equation is

$$\omega = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ a_1 & 0 & b_3 & -b_2 \\ a_2 & -b_3 & 0 & b_1 \\ a_3 & b_2 & -b_1 & 0 \end{pmatrix} \equiv i a_i K_i + i b_i J_i$$

where the matrices J and K are defined by the last equivalence, and the factor of i is just a convention. It is easy to show that the matrices J_i are the infinitesimal rotation generators and the K_i matrices are the infinitesimal boost generators. For example, K_2 is the matrix

$$(K_2)^\mu{}_\nu = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and under an infinitesimal boost in the y direction, the coordinates change by

$$\delta x^\mu = i\epsilon (K_2)^\mu{}_\nu x^\nu \ .$$

Since all the possible infinitesimal Lorentz transformations are either rotations or boosts, we may conclude that all Λs satisfying (2) are some combinations of boosts and rotations.[2]

The Lorentz Algebra

By explicit calculation, we can find the commutation relations of the Lorentz generators. We have

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}$$

where ϵ^{ijk} is defined to be 1 for $(ijk) = (123), (231), (312)$ and -1 for $(ijk) = (213), (321), (132)$.

These commutation relations will be important for determining the possible ways the field variables can transform under Lorentz transformations. To do this, we'll want to determine all possible ways to find matrices (not necessarily 4 by 4) that have the same commutation relations as these.

[2] Again, to be precise, we should say that this is true for any Lorentz transformation that can be deformed smoothly to the identity transformation. The most general transformation is some combination of these plus a possible parity transformation and/or time reversal.

Exercises

a) Write out all the terms in the expression $x^\mu \partial_\mu \phi$.

b) Write out all the terms in the expression $x_\mu p^\mu$.

c) The scalar and vector potentials in electromagnetism form a four-vector field

$$A^\mu(t, \vec{x}) = \begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

under Lorentz transformations. Starting with the potential

$$A^\mu(t, \vec{x}) = \begin{pmatrix} Ey/c \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

(i.e. a constant electric field), what is the potential after a boost by velocity v in the x direction (last equation on page 1 of the notes)?