

# 1 A Brief Review of Quantum Mechanics

Before starting to learn about quantum field theory, it's probably a good idea to make sure you have a solid idea of what quantum mechanics is all about.

## 1.1 Classical Mechanics

To start, let's recall how physical systems are described classically, so that we can contrast with the quantum mechanical description.

The configuration of a physical system at a particular time is described by a set of variables. These could be coordinates for some number of particles  $\vec{x}_i(t)$ , angles describing the orientation of a rigid body  $\theta_i(t)$ , functions describing the displacement of a string as a function of position along the string  $\phi(x, t)$ , etc... . Sometimes it will be useful to describe this information in an abstract way by some *generalized coordinates*  $q_\alpha(t)$ , where  $\alpha$  could either run over a finite number of values (as in the case of the coordinates for a particle), or an infinite number of values (as in the case where  $\alpha$  describes the locations along a string).

To completely specify the state of a classical system at a particular time, we typically need to know the coordinates  $q_\alpha(t)$  at that time and also the velocities  $\dot{q}_\alpha(t)$ , i.e. how quickly they are changing at that particular time.<sup>1</sup> All other physical quantities, such as the accelerations, energies, momenta, etc... can be derived from this information either because they are simply defined as functions of the coordinates and velocities, or because some physical law tells us what they must be. In particular, the laws of classical physics can be used to derive equations of motion that determine the configuration of the system at any future time from the configuration (coordinates and velocities) at some initial time:

$$q_\alpha(t = t_0) , \dot{q}_\alpha(t = t_0) \quad \Longrightarrow_{EOM} \quad q_\alpha(t)$$

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<sup>1</sup>In Maxwell's equations for electromagnetism, we do not need the time derivatives of the electric and magnetic fields to determine their future evolution. However, when the information is given in terms of scalar and vector potentials, both the potentials and their time derivatives are needed.

## 1.2 Quantum Mechanics

In quantum mechanics, we often speak of “quantizing” a particular physical system. What does this really mean?

First, it is important to emphasize that as far as we know, all physical systems are quantized; that is, they are governed fundamentally by the rules of quantum mechanics. In particular, the description in terms of classical mechanics is at best an approximation that works well in certain situations (e.g. if the system is macroscopic and carries enough energy).

So “quantizing” the system really means moving to the more precise quantum mechanical description that will have a wider range of validity than the classical description.

Though classical physics gives a very good approximation to quantum mechanics in many situations, it is crucial to realize that the quantum mechanical description does not just involve some small (or large) corrections to the classical equations of motion but rather a completely different conceptual and mathematical framework for the description of nature.

In a classical description, all physical quantities have definite values in every configuration. Once we know  $q_\alpha(t)$  and  $\dot{q}_\alpha(t)$ , we can calculate energy, momentum, acceleration, and so forth. In quantum mechanics, for any given physical quantity (or *observable*), only special states (the *eigenstates* for that observable) will have definite values for that observable (these values are the *eigenvalues*). For more general states, a measurement of some observable will give a result that cannot be predicted, even if we have all possible information about the state: *general states do not have definite values for physical quantities*. The only thing we can predict is the probability for various outcomes of the measurement.

Mathematically, it turns out that the quantitative description of quantum mechanics fits into the framework of linear algebra. Configurations of a physical system are described by *vectors* in a complex vector space with an inner product (or Hilbert space, which is a kind of vector space with an infinite number of basis vectors). This means that given any two configurations, there is another configuration given by the sum of those configurations with arbitrary complex coefficients. This sum is known as a *quantum superposition*. The eigenstates for a particular observable correspond to some complete, orthogonal set of unit basis vectors for the Hilbert space. Any other vector in the Hilbert space can therefore be written as a superposition of the basis vectors (i.e. any state which is not an eigenstate can be written

as a quantum superposition of eigenstates). The coefficients in the superposition determine the probabilities for the various outcomes of a measurement of that observable: if the eigenstate vectors are denoted by  $|\lambda_i\rangle$  and we can decompose the vector corresponding to a general state  $|\psi\rangle$  as

$$|\psi\rangle = \sum_i c_i |\lambda_i\rangle$$

then the probability of finding  $\lambda_i$  when we measure the observable is proportional to  $|c_i|^2$ . If  $\lambda_i$  is the result of our measurement, the state of the system immediately after the measurement becomes  $|\lambda_i\rangle$ , so that an immediately repeated measurement will definitely give the same result.

For some observables (e.g. position), the set of possible values is continuous. In that case, a general quantum superposition is written as an integral

$$\int dx \psi(x) |x\rangle .$$

In this case, the coefficients in the quantum superposition make up a function  $\psi(x)$  which we call the wavefunction (in this case, the position-space wavefunction). The wavefunction (or equivalently, the vector in the Hilbert space) carries all information about the state of the system at a particular time.

### 1.2.1 Quantum uncertainty

A key feature of quantum mechanics is that the basis vectors corresponding to different observables do not generally coincide. For example, eigenstates of position are not eigenstates of momentum, but quantum superpositions of states with different momentum. This leads to the uncertainty principle in quantum mechanics.

### 1.2.2 Observables and operators

An operator is a linear map from a Hilbert space to itself, taking vectors into other vectors. In quantum mechanics, any physical observable can be naturally associated with an operator, which takes the eigenvectors for that observable to

$$\mathcal{O}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$$

This completely defines the operator, since linearity implies that the action on any general vector must be

$$\mathcal{O} \sum_i c_i |\lambda_i\rangle = \sum_i c_i \lambda_i |\lambda_i\rangle$$

Operators corresponding to physical observables have the special property of being Hermitian ( $\mathcal{O}^\dagger = \mathcal{O}$ ), related to the fact that all of their eigenvalues are real.

### 1.2.3 Symmetries

Another class of operators in quantum mechanics are the operators that carry out physical transformations such as translations, rotations, or time translations on states. These operators must be *unitary* (they satisfy  $\mathcal{O}^\dagger \mathcal{O} = 1$ ) in order that  $\mathcal{O}|\psi\rangle$  has norm 1 if  $|\psi\rangle$  has norm 1.

Related to any continuous physical transformation is an infinitesimal version of that transformation. For example, suppose that  $\mathcal{T}(a)$  is the operator that translates states by an amount  $a$  in the  $x$  direction. The small change in a state under an infinitesimal translation is:

$$\delta|\psi\rangle = \mathcal{T}(\delta x)|\psi\rangle - |\psi\rangle \approx \delta x \mathcal{T}'(0)|\psi\rangle \quad (1)$$

If we define  $p = i\hbar \mathcal{T}'(0) = i\hbar \lim_{\epsilon \rightarrow 0} (\mathcal{T}(\epsilon) - 1)/\epsilon$ , we can show that  $p$  is a Hermitian operator from the fact that  $\mathcal{T}$  is a unitary operator. The  $\hbar$  here is just a convention.

So generally, we can associate a Hermitian operator to any infinitesimal transformation. But we already said that Hermitian operators were associated with physical observables. This leads to a fundamental fact about quantum mechanics: to any infinitesimal transformation, we can associate a physical observable. We'll learn later that if the infinitesimal transformation is a symmetry, then the observable will be a conserved quantity.

So what is the physical observable related to the infinitesimal translation operator  $p$ ? It's the conserved quantity associated with spatial translation invariance, namely momentum (the factor of  $\hbar$  above was chosen to agree with the conventional definition). From the equation (1) above and our definition of  $p$ , we thus have that under an translation in the  $x$  direction by an amount  $\delta x$ , we have

$$\delta|\psi\rangle = \delta x p / (i\hbar) |\psi\rangle . \quad (2)$$

where  $p$  is the momentum operator. From this, we can derive the fundamental relation<sup>2</sup>

$$[x, p] = i\hbar .$$

For more general quantum systems, a similar commutation relation holds between any generalized coordinate and the corresponding momentum.

In a similar way, the infinitesimal time translation operator is proportional to the energy operator  $H$ . Thus, under an infinitesimal translation in time, we have:

$$\delta|\psi\rangle = H/(i\hbar)|\psi\rangle\delta t .$$

Rearranging, we see that this is exactly the Schrödinger equation:

$$i\hbar \frac{\delta|\psi\rangle}{\delta t} = H|\psi\rangle .$$

#### 1.2.4 Time evolution

Understanding properties of the energy eigenstates is a key step in determining the time evolution of a system. From the Schrodinger equation, it is easy to see that if  $|\psi(t = 0)\rangle$  is an energy eigenstate  $|E\rangle$  with energy  $E$ , then<sup>3</sup>

$$|\psi(t)\rangle = e^{-iEt/\hbar}|E\rangle .$$

Given any other initial state  $|\psi(0)\rangle$ , if we can write it as a linear combination of energy eigenstates,

$$|\psi(t = 0)\rangle = \sum_n c_n |E_n\rangle .$$

Then the time-evolved state must (by linearity of the Schrodinger equation) be

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |E_n\rangle .$$

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<sup>2</sup>As an intermediate step, if we define  $\psi(x) = \langle x|\psi\rangle$ , we then find that

$$\begin{aligned} \langle x|p|\psi\rangle &= (i\hbar)\langle x|\delta|\psi\rangle/\delta x \\ &= (i\hbar)\lim_{\epsilon\rightarrow 0}(\langle x|T(\delta x)|\psi\rangle - \langle x|\psi\rangle)/\delta x \\ &= (i\hbar)\lim_{\epsilon\rightarrow 0}(\psi(x - \delta x) - \psi(x))/\delta x \\ &= -i\hbar\psi'(x) . \end{aligned} \tag{3}$$

<sup>3</sup>Here, we are assuming that  $H$  is independent of time.

Thus, given a quantum system, the general problem of time evolution will be solved if we can determine the energy eigenvalues and how to represent any arbitrary state of interest in terms of energy eigenstates.<sup>4</sup>

## 2 The harmonic oscillator

As an example, suppose we want to understand the quantum physics of the harmonic oscillator, whose classical configurations are described by a single coordinate  $x(t)$ , governed by equations of motion

$$\ddot{x} = -\omega^2 x$$

arising from an action

$$S = \int dt \left\{ \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right\}$$

and whose energy is given by

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega^2 x^2 .$$

A first step in understanding the quantum behavior of the system is to determine what the possible energy eigenvalues are.<sup>5</sup>

We start by rewriting the classical expression for energy in terms of position and momentum:<sup>6</sup>

$$E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

Reinterpreting this as a quantum mechanical operator (where  $p$  and  $x$  are the position and momentum operators) gives us the quantum Hamiltonian

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{x}^2$$

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<sup>4</sup>For example, if we want to understand the time-evolution of some arbitrary position space wavefunction, we would need to understand how to write the energy eigenstates in terms of position space wavefunctions.

<sup>5</sup>If we want, we could then figure out the position space wavefunctions for any energy eigenstate. With this information, we would be able to predict the time evolution of any initial wavefunction.

<sup>6</sup>Here, the momentum is  $p = m\dot{x}$  as usual, but more generally, the momentum corresponding to a variable  $q$  is determined in terms of the Lagrangian by  $p = \delta L / \delta \dot{q}$ .

The only other information we need to determine the energy eigenvalues is the basic commutation relation between position and momentum operators:<sup>7</sup>

$$[\mathbf{x}, \mathbf{p}] = i\hbar .$$

To find the energy eigenvalues, we can use the fancy formal trick of defining creation and annihilation operators:

$$\begin{aligned} a &= \sqrt{\frac{1}{2\hbar m\omega}}(m\omega x + ip) \\ a^\dagger &= \sqrt{\frac{1}{2\hbar m\omega}}(m\omega x - ip) \end{aligned} .$$

With these definitions, we can check that

$$[a, a^\dagger] = 1 \qquad [H, a] = -\hbar\omega a \qquad [H, a^\dagger] = \hbar\omega a^\dagger .$$

Also, the Hamiltonian can now be written as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) . \tag{4}$$

So how do we find the energy eigenvalues? By definition, acting with the Hamiltonian (i.e. the energy operator) on a state  $|E_i\rangle$  that is an energy eigenvalue gives

$$H|E_i\rangle = E_i|E_i\rangle .$$

Now, suppose  $|E_i\rangle$  is an eigenvector with eigenvalue  $E_i$ . Now, let's consider the state  $a|E_i\rangle$  and see what happens when we act with  $H$ :

$$\begin{aligned} H(a|E_i\rangle) &= aH|E_i\rangle - \hbar\omega a|E_i\rangle \\ &= (E_i - \hbar\omega)a|E_i\rangle \end{aligned}$$

There are two possibilities here: either  $a|E_i\rangle$  is another eigenstate of the Hamiltonian with a lower energy eigenvalue ( $E_i - \hbar\omega$ ), or  $a|E_i\rangle$  is just equal to zero (i.e. it is not a state at all). Assuming that there is some minimum energy state for the harmonic oscillator, we can conclude that by starting with any energy eigenstate and acting repeatedly with the annihilation operator

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<sup>7</sup>This is equivalent to the statement that the momentum operator is the generator of translations.

$a$ , we will keep getting new eigenstates with lower and lower energy, but eventually we must find a state  $|E_0\rangle$  such that  $a|E_0\rangle = 0$ .

We will show below that the condition  $a|E_0\rangle = 0$  uniquely defines a state. This means that starting with any energy eigenstate and acting with  $a$ , we will always end up in the state  $E_0$ . On the other hand, we can show that if  $a^n|E_i\rangle = |E_0\rangle$  that  $(a^\dagger)^n|E_0\rangle$  is proportional to  $|E_i\rangle$ . Thus, all possible eigenstates can be obtained by acting with  $a^\dagger$  repeatedly on  $|E_0\rangle$ .

To find the actual energies, we note that (4) implies that the energy of  $|E_0\rangle$  is  $E_0 = \frac{1}{2}\hbar\omega$ . Further, we can check that acting with  $a^\dagger$  raises the energy by  $\hbar\omega$  (we have already shown that acting with  $a$  lowers the energy by this amount). Thus, the possible energies are  $E_n = \hbar\omega(n + \frac{1}{2})$ . It is conventional to label the eigenstate with energy  $E_n$  by the shorthand  $|n\rangle$ , normalized so that  $\langle n|n\rangle = 1$ . With this convention, we can show that

$$\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ a|n\rangle &= \sqrt{n}|n-1\rangle \\ H|n\rangle &= \hbar\omega(n + \frac{1}{2})|n\rangle \end{aligned}$$

## 2.1 Loose end

To tie things up, we should go back and show that the state satisfying  $a|\psi\rangle = 0$  is unique. To do this, it is convenient to write this equation (a vector equation) in terms of its position space components. The component of a state vector  $|\psi\rangle$  along the basis vector  $|x\rangle$  can be calculated by taking the inner product of  $|\psi\rangle$  with the unit vector:

$$\langle x|\psi\rangle \equiv \psi(x).$$

Thus, the component of the vector equation above along the  $|x\rangle$  direction is:

$$\langle x|a|\psi\rangle = 0$$

Putting in the definition of  $a$ , we get

$$\langle x|(m\omega x + ip)|\psi\rangle = 0$$

Now, we have  $\langle x|x|\psi\rangle = x\psi(x)$  and  $\langle x|p|\psi\rangle = -i\hbar\psi'(x)$  (using (3)). Thus,  $\psi(x)$  satisfies the differential equation

$$m\omega x\psi(x) + \hbar\psi'(x) = 0.$$



This has a solution

$$\psi(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

where the constant  $A$  is fixed up to an overall phase by demanding that  $\langle\psi|\psi\rangle = 1$ . Vectors which differ only by an overall phase are physically equivalent in quantum mechanics, so we have shown that the state satisfying  $a|\psi\rangle = 0$  is unique (and found its position space wavefunction).