Creation and annihilation operators for fermions

Consider a quantum mechanical system of non-interacting fermions. Suppose that the single-particle energy eigenstates of the system are described by wavefunctions $\psi_p(x)$ (if there is no potential, p might label the momentum).

Q: If we have a system of two particles, with one particle in state p and the other particle in state q, what is the wavefunction $\psi(x_1, x_2)$ for the state $|(p,q)\rangle$ of the two particles in terms of the single-particle wavefunctions? *Hint: how would your answer be different if we were talking about bosons?*

$$\psi(x_1,x_2) = \psi_p(x_1)\psi_q(x_2) - \psi_p(x_2)\psi_q(x_1)$$

Q: Based on your answer to the previous question, how is the state $|(q,p)\rangle$ (obtained by swapping the two particles) related to the state $|(p,q)\rangle$?

$$|(q,p)\rangle = -|(p,q)\rangle$$

Q: Now suppose we have a quantum field theory system that can describe arbitrarily many of these non-interacting particles. In this theory, there will be an operator $a_p^\dagger$ that creates a particle in the state p. In terms of these operators, how do we write the state $|(p,q)\rangle$?

$$|(p,q)\rangle = a_p^\dagger a_q^\dagger |0\rangle$$

(we could have also picked $|(p,q)\rangle = a_q^\dagger a_p^\dagger |0\rangle$)

Q: Based on your previous two answers, how is $a_p^\dagger a_q^\dagger$ related to $a_q^\dagger a_p^\dagger$?

$$a_p^\dagger a_q^\dagger = -a_q^\dagger a_p^\dagger$$

Q: If we take $p = q$, what does the previous relation imply about the state $a_p^\dagger a_p^\dagger |0\rangle$?

$$a_p^\dagger a_p^\dagger + a_p^\dagger a_p^\dagger = 0 \qquad \therefore \quad a_p^\dagger a_p^\dagger |0\rangle = 0$$

Q: On the subspace of states with basis $|0\rangle$ and $|p\rangle = a_p^\dagger|0\rangle$, how is the operator $a_p^\dagger$ represented (as a two by two matrix)?

Let $|0\rangle \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad a_p^\dagger |0\rangle = |p\rangle$

$|p\rangle \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad a_p^\dagger |p\rangle = a_p^\dagger a_p^\dagger |0\rangle = 0$

$\therefore \quad a_p^\dagger$ is represented by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Q: In terms of this matrix, calculate the matrix $a_p a_p^\dagger + a_p^\dagger a_p$ and the matrix $a_p a_p^\dagger - a_p^\dagger a_p$. Which one is proportional to the identity matrix?

$$a_p \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a_p a_p^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad a_p^\dagger a_p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$a_p a_p^\dagger + a_p^\dagger a_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix

$a_p a_p^\dagger - a_p^\dagger a_p = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ not $\propto \mathbb{1}$.

2 For $S = \int \left(\frac{1}{2}\partial_\mu \bar{\psi}\partial^\mu \psi - \frac{1}{2}m^2 \bar{\psi}\psi\right) d^4x$, we can write everything out in components by defining

$$\psi_\alpha = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ a_3 + ib_3 \\ a_4 + ib_4 \end{pmatrix}$$

Then:

$$\bar{\psi} = \psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (a_3 - ib_3, a_4 - ib_4, a_1 - ib_1, a_2 - ib_2)$$

$$S = \int d^4x \, \{ \partial_\mu a_1 \partial^\mu a_3 + \partial_\mu a_2 \partial^\mu a_4 + \partial_\mu b_1 \partial^\mu b_3 + \partial_\mu b_2 \partial^\mu b_4 \\ - m^2 a_1 a_3 - m^2 a_2 a_4 - m^2 b_1 b_3 - m^2 b_2 b_4 \}$$

To diagonalize, we can define

$$a_1 = \tfrac{1}{2}(x_1 + y_1) \qquad b_1 = \tfrac{1}{2}(x_3 + y_3)$$
$$a_3 = \tfrac{1}{2}(x_1 - y_1) \qquad b_3 = \tfrac{1}{2}(x_3 - y_3)$$
$$a_2 = \tfrac{1}{2}(x_2 + y_2) \qquad b_2 = \tfrac{1}{2}(x_4 + y_4)$$
$$a_4 = \tfrac{1}{2}(x_2 - y_2) \qquad b_4 = \tfrac{1}{2}(x_4 - y_4)$$

With the change of variables,

$$S = \int d^4x \, \{ \tfrac{1}{2}\partial_\mu x^i \partial^\mu x^i - \tfrac{1}{2} m^2 x^i x^i \\ - \tfrac{1}{2}\partial_\mu y^i \partial^\mu y^i + \tfrac{1}{2} m^2 y^i y^i \}$$

The terms with y all have the wrong sign, and lead to both kinetic & potential energy unbounded below.

3

a) If $(m\gamma^0 - m)u(0) = 0$, then

$$
\begin{aligned}
(p_\mu\gamma^\mu - m)u(\vec{p}) &= (p_\mu\gamma^\mu - m)\Lambda_{\frac{1}{2}}(\vec{p})u(0) \\
&= \Lambda_{\frac{1}{2}}(\vec{p})\left[p_\mu \Lambda_{\frac{1}{2}}^{-1}(\vec{p})\gamma^\mu\Lambda_{\frac{1}{2}}(\vec{p}) - m\right]u(0) \\
&= \Lambda_{\frac{1}{2}}(\vec{p})\left[p_\mu \Lambda^\mu{}_\nu(\vec{p})\gamma^\nu - m\right]u(0) \\
&= \Lambda_{\frac{1}{2}}(\vec{p})\left[m\gamma^0 - m\right]u(0) \\
&= 0
\end{aligned}
$$

Here, we have used $\Lambda p_0 = p \Rightarrow \Lambda^T \eta p = \eta p_0$
$\Rightarrow p_\mu \Lambda^\mu{}_\nu = (p_0)_\nu$

where $p_0 = (m,0,0,0)$.

We have:

$$
\begin{aligned}
\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) &= \sum_s \Lambda_{\frac{1}{2}}(p)\, u_s(0)\, u_s^\dagger(0)\, \Lambda_{\frac{1}{2}}^\dagger(p)\gamma^0 \\
&= \Lambda_{\frac{1}{2}}(p)\cdot m\sum_s \begin{pmatrix} \xi_s\xi_s^\dagger & \xi_s\xi_s^\dagger \\ \xi_s\xi_s^\dagger & \xi_s\xi_s^\dagger \end{pmatrix} \Lambda_{\frac{1}{2}}^\dagger(p)\gamma^0 \\
&= \Lambda_{\frac{1}{2}}(p)\cdot m\,(\gamma^0 + \mathbb{1})\,\Lambda_{\frac{1}{2}}^\dagger(p)\gamma^0 \qquad \text{used } \Lambda_{\frac{1}{2}}^{-1}\gamma^0 = \gamma^0\Lambda_{\frac{1}{2}}^\dagger \\
&= \Lambda_{\frac{1}{2}}(p)\, m\Lambda_{\frac{1}{2}}^{-1}\gamma^0\gamma^0 + m\,\Lambda_{\frac{1}{2}}(p)\gamma^0\Lambda_{\frac{1}{2}}^{-1}(p) \\
&= m + m\,\Lambda^{0}{}_{\nu}^{-1}\gamma^\nu \\
&= m + \gamma_\nu \Lambda^\nu{}_\alpha p_0^\alpha \qquad (p_0 = (m,0,0,0)) \\
&= m + \gamma_\nu p^\nu \\
&= \not{p} + m
\end{aligned}
$$