# PROBLEM SET 5 SOLUTIONS:

① For a 3+1 dimensional scalar field theory, the field operator can be expressed in terms of the creation and annihilation operators as:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left( a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

The operator version of $\dot{\phi}$, $\pi(x)$ can be expressed as:

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}}\, i \left( -a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

The momentum operator is

$$\hat{P}_x = - \int d^3x\, \pi(\vec{x}) \cdot \partial_x \phi(\vec{x})$$

$$= - \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{E_p}{\sqrt{2E_p}} i \left( a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$
$$\cdot \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \left( i q_x\, a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}} (-iq_x)\, e^{-i\vec{q}\cdot\vec{x}} \right)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \sqrt{\frac{E_p}{E_q}} \Big( -q_x\, a_{\vec{p}} a_{\vec{q}} (2\pi)^3 \delta(\vec{p}+\vec{q})$$
$$+ q_x\, a^\dagger_{\vec{p}} a_{\vec{q}} (2\pi)^3 \delta(\vec{p}-\vec{q})$$
$$+ q_x\, a_{\vec{p}} a^\dagger_{\vec{q}} (2\pi)^3 \delta(\vec{p}-\vec{q})$$
$$- q_x\, a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} (2\pi)^3 \delta(\vec{p}+\vec{q}) \Big)$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left\{ a_{\vec{p}} a_{-\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}} \right\} p_x$$

$$= \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}} \cdot p_x + \int \frac{d^3p}{(2\pi)^3} p_x \cdot \underbrace{\left\{ \text{terms even under } (p_x \leftrightarrow -p_x) \right\}}$$

$$= \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}} \cdot p_x$$

these must be zero after $p_x$ integration

We have $\hat{P}_x \, a^\dagger_{\vec{p}} |0\rangle$

$$= \left\{ \int \frac{d^3q}{(2\pi)^3} a^\dagger_{\vec{q}} a_{\vec{q}} \, q_x \right\} a^\dagger_{\vec{p}} |0\rangle$$

$$= \int \frac{d^3q}{(2\pi)^3} q_x \, a^\dagger_{\vec{q}} \, a_{\vec{q}} a^\dagger_{\vec{p}} |0\rangle$$

$$= \int \frac{d^3q}{(2\pi)^3} q_x \, a^\dagger_{\vec{q}} \left\{ \cancelto{0}{a^\dagger_{\vec{p}} a_{\vec{q}}} + (2\pi)^3 \delta(\vec{p}-\vec{q}) \right\} |0\rangle$$

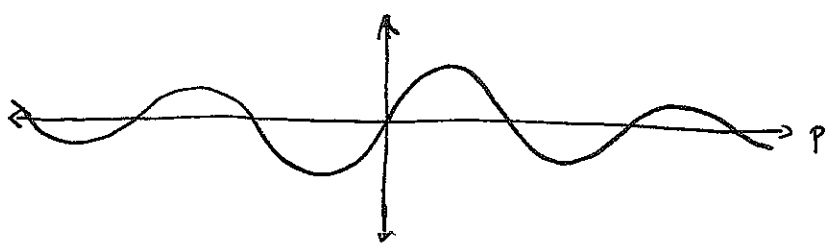
$$= p_x \left\{ a^\dagger_{\vec{p}} |0\rangle \right\} \quad , \text{ as desired.}$$

② We have: $[a_p, a_q^\dagger] = (2\pi)\delta(p-q)$, $[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0$. Thus:

$$
\begin{aligned}
&[\phi(x_1,t_1), \phi(x_2,t_2)] \\
&= \int \frac{dp}{(2\pi)} \frac{1}{\sqrt{2E_p}} \int \frac{dq}{(2\pi)} \frac{1}{\sqrt{2E_q}} \Big\{ \big[a_p e^{-ip\cdot x_1}, a_q^\dagger e^{iq\cdot x_2}\big] \\
&\qquad\qquad + \big[a_p^\dagger e^{ip\cdot x_1}, a_q e^{-iq\cdot x_2}\big] \Big\} \\
&= \int \frac{dp}{(2\pi)} \int \frac{dq}{(2\pi)} \frac{1}{\sqrt{4E_pE_q}} \Big\{ \delta(p-q)\, e^{-ip\cdot x_1 + iq\cdot x_2} \cdot (2\pi) \\
&\qquad\qquad - (2\pi)\,\delta(p-q)\, e^{ip\cdot x_1 - iq\cdot x_2} \Big\} \\
&= \int \frac{dp}{(2\pi)} \frac{1}{2E_p} \Big\{ e^{-ip\cdot(x_1-x_2)} - e^{ip\cdot(x_1-x_2)} \Big\} \\
&= \int \frac{dp}{(2\pi)} \frac{1}{2\sqrt{m^2+p^2}} \Big\{ e^{-iE_p(t_1-t_2) + ip(x_1-x_2)} - e^{+iE_p(t_1-t_2) - ip(x_1-x_2)} \Big\}
\end{aligned}
$$

For $t_1 = t_2$, this gives:

$$
\int \frac{dp}{(2\pi)} \underbrace{\frac{2i}{2\sqrt{m^2+p^2}}}_{\text{even}} \underbrace{\sin\big(p(x_1-x_2)\big)}_{\text{odd.}} = 0
$$



③ We have: $\langle 0|\phi(\vec{x},0)\,\phi(0,0)|0\rangle$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\langle 0|\,a_p e^{i\vec{p}\cdot\vec{x}}\,a_q^\dagger\,|0\rangle$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\, e^{i\vec{p}\cdot\vec{x}}\,(2\pi)^3\delta^3(\vec{p}-\vec{q})$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\cdot e^{i\vec{p}\cdot\vec{x}}$$

To evaluate this, it is convenient to write

$$\frac{1}{E_p} = \frac{1}{\sqrt{m^2+p^2}} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{d\alpha}{\sqrt{\alpha}}\, e^{-\alpha(m^2+p^2)}$$

↖ general trick: $\frac{1}{A^s} = \frac{1}{\Gamma(s)}\int_0^\infty dt\, e^{-tA}\, t^{s-1}$

This gives:

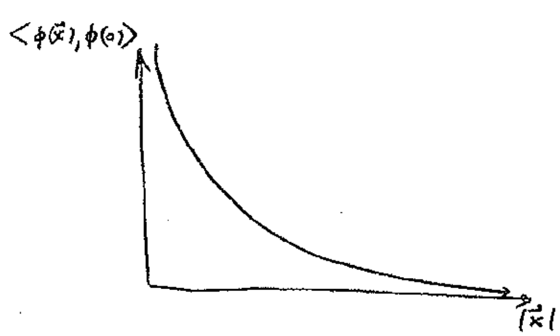
$$\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{d\alpha}{\sqrt{\alpha}}\,e^{-\alpha m^2}\int\frac{d^3p}{(2\pi)^3}\,e^{-\alpha p^2+i\vec{p}\cdot\vec{x}}$$

complete the square & do the Gaussian integral.

$$= \frac{1}{2\sqrt{\pi}}\,\frac{1}{(2\pi)^3}\,\pi^{\frac{3}{2}}\int_0^\infty\frac{d\alpha}{\alpha^2}\,e^{-\alpha m^2-\frac{x^2}{4\alpha}}$$

$$= \frac{m}{4\pi^2|x|}K_1(mx)$$

This falls of exponentially for large $x$, with decay constant $\propto \frac{1}{m}$. Thus, the larger the mass, the shorter the correlations.



④ a) The Lagrangian density $\mathcal{L}(x)$ must be built from $A_\mu$ & its derivatives in such a way that all vector indices are paired up (using $\eta_{\mu\nu}$ or $\varepsilon$ tensors). We can have no more than two derivatives if $A$ and $\dot{A}$ determine the classical evolution. Also, we can have no more than 2 $A$'s for linear eq. of. motion. Ignoring terms which are obviously total derivatives, we can have:

$$\overset{①}{A_\mu A^\mu}\,,\ \overset{②}{\partial_\mu A^\mu \partial_\nu A^\nu}\,,\ \overset{③}{\partial_\mu A_\nu \partial^\mu A^\nu}\,,\ \overset{④}{A_\mu \partial^\mu \partial^\nu A_\nu}\,,\ \overset{⑤}{A_\mu \partial^\nu \partial_\nu A^\mu}$$

⑥: $\partial_\mu A^\nu \partial_\nu A^\mu$

However, some of these differ only by total derivatives (i.e. are equivalent after integration by parts. We see that:

$$\partial_\mu (A^\mu \partial_\nu A^\nu) = \partial_\mu A^\mu \partial_\nu A^\nu + A^\mu \partial_\mu \partial_\nu A^\nu$$

$$\partial_\mu (A^\nu \partial_\nu A^\mu) = \partial_\mu A^\nu \partial_\nu A^\mu + A^\nu \partial_\mu \partial_\nu A^\mu$$

$$\partial_\mu (A^\nu \partial^\mu A_\nu) = \partial_\mu A^\nu \partial^\mu A_\nu + A^\nu \partial_\mu \partial^\mu A_\nu$$

Thus: ④ is equivalent to ②
⑥ is equivalent to ④ and therefore to ②.
⑤ is equivalent to ③

A minimal set of terms is:

$$S = \int d^3x \left\{ C_1 A_\mu A^\mu + C_2 \partial_\mu A_\nu \partial^\mu A^\nu + C_3 \partial_\mu A^\nu \partial_\nu A^\mu \right\}$$

b) Substituting $A_\mu \to A_\mu + \partial_\mu \lambda$, we find:

$$S \to S + \int d^3x \Big\{ 2C_1 A^\mu \partial_\mu \lambda + C_2 \partial_\mu \partial_\nu \lambda \partial^\mu A_\nu + C_3 \partial_\mu \partial^\nu \lambda \partial_\nu A^\mu$$
$$+ C_2 \partial_\mu A_\nu \partial^\mu \partial_\nu \lambda + C_3 \partial_\mu A^\nu \partial_\nu \partial^\mu \lambda \Big\} + O(\lambda^2)$$

The first term here has one A, one $\lambda$, and one derivative, so there is no other term that can cancel it. We must have $C_1 = 0$. The rest of the terms have 3 derivatives, one A, and one $\lambda$. After integration by parts, we can write this set of terms as:

$$\int d^3x \; A_\mu \partial^\mu \partial_\nu \partial^\nu \lambda \left\{ -2C_2 - 2C_3 \right\}$$

This vanishes if $C_3 = -C_2$. Thus, the most general action with the symmetry specified is

$$S = \int d^3x \; C_2 \left( \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu \right)$$

c) In 2+1 dimensions, we can have

$$\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$