$\boxed{\text{PROBLEM SET 4 SOLUTIONS}}$

1

b) $\partial_\mu \phi \partial^\mu \phi = \frac{1}{c^2}(\partial\phi/\partial t)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2$

c) We have:

$$\tilde{A}^\mu(\Lambda x) = \Lambda^\mu{}_\nu A^\nu(x)$$

OR: $$\tilde{A}^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

With $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}$ we have $\Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \vec{0} \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}$

Then $\Lambda^{-1}x = \begin{pmatrix} \gamma ct + \gamma\beta x \\ \gamma\beta \cdot ct + \gamma x \\ y \\ z \end{pmatrix}$ note y is unchaged

and $$\tilde{A}^\mu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Ey/c \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\gamma E y}{c} \\ -\frac{\gamma\beta E y}{c} \\ 0 \\ 0 \end{pmatrix}$$

Note that we have a const electric & magnetic field after the boost

(2)

a) The angular momentum operator $\overset{J_x}{S_x}$ gives the infinitesimal change in a state after a rotation about the x axis.

we have:

$$\delta|\phi\rangle = \delta\theta \cdot \frac{J_x}{i\hbar}|\phi\rangle \qquad (*)$$

$$= -\frac{i}{\hbar}\delta\theta\,\frac{1}{2}(J_+ + J_-)|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$\Rightarrow \boxed{\delta|\phi\rangle = -\frac{i}{2}\delta\theta \cdot |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle}$$

b) For a 45° rotation, we need to solve/integrate the differential equation (*):

$$\frac{\delta|\phi\rangle}{\delta\theta} = \frac{J_x}{i\hbar}|\phi\rangle$$

This gives:

$$|\phi_{45°}\rangle = e^{\frac{J_x}{i\hbar}\cancel{\delta\theta}\frac{\pi}{4}}|\phi\rangle$$

(3) Starting with $S = \int dt \int_0^L dx \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right\}$, we can change variables to Fourier modes by writing:

$$\phi(x) = \underbrace{\sum_{n=0}^{\infty} \phi_n^c \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} \phi_n^s \sin\left(\frac{2n\pi x}{L}\right)}_{\text{real}}$$

OR

$$\phi(x) = \sum_{n=-\infty}^{\infty} \underbrace{\phi_n}_{\text{complex}} e^{\frac{2\pi i n x}{L}} \qquad \text{with} \qquad \phi_{-n}^* = \phi_n$$

to ensure that ϕ is real

(we take these forms to ensure $\phi(x+L) = \phi(x)$)

We can solve the problem using either choice of variables. With the sines + cosines, we find that:

$$E = \int dt \int_0^L dx \left\{ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\partial_x \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

$$= \frac{L}{4} \sum_{n=0}^{\infty} (\dot{\phi}_n^c)^2 + \left[\left(\frac{2\pi n}{L} \right)^2 + m^2 \right] (\phi_n^c)^2$$

$$+ \frac{L}{4} \sum_{n=1}^{\infty} (\dot{\phi}_n^s)^2 + \left[\left(\frac{2\pi n}{L} \right)^2 + m^2 \right] (\phi_n^s)^2$$

So ϕ_n^c and ϕ_n^s all correspond to harmonic oscillators with

$$M = \frac{L}{2} \qquad \omega_n = \sqrt{m^2 + \left(\frac{2\pi n}{L} \right)^2}$$

We then define:

$$P_n^s = M \dot{\phi}_n^s = \frac{L}{2} \dot{\phi}_n^s \qquad P_n^c = M \dot{\phi}_n^c = \frac{L}{2} \dot{\phi}_n^c$$

To convert classical observables to quantum operators, we write everything in terms of $\phi_n^{c,s}$ and $p_n^{c,s}$ then promote these to operators obeying $[\phi_n^{c,s}, p_n^{c,s}] = i\hbar$.

For this problem, we're interested in finding momentum eigenstates. Using the standard procedure, we can derive the momentum operator by considering the variation of S under $\delta\phi = -\varepsilon(x,t)\,\phi'$. This gives:

$$\delta S = -\int dt \int dx\ \dot{\varepsilon}\left(\dot{\phi}\phi'\right) + \varepsilon'\left(-\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\phi')^2 + \tfrac{1}{2}(m^2\phi^2)\right)$$

The momentum density is then $\dot{\phi}\phi'$ so the net momentum is

$$P = -\int_0^L dx\ \dot{\phi}\phi' .$$

To write the quantum operator, we first change variables to modes:

$$P = -\sum_{n=1}^{\infty} n\pi\left(\dot{\phi}_n^c \phi_n^s - \dot{\phi}_n^s \phi_n^c\right)$$

$$= \frac{2\pi}{L}\sum_{n=1}^{\infty} n\left(-p_n^c \phi_n^s + p_n^s \phi_n^c\right)$$

To analyze the single particle states, we define cr. & annih. ops as usual by:

$$\phi_n^c = \left(a_n^c + a_n^{c\dagger}\right)\cdot\frac{1}{\sqrt{2M\omega_n}} = \frac{1}{\sqrt{L\omega_n}}\left(a_n^c + a_n^{c\dagger}\right) \qquad \text{(we are using } \hbar = 1\text{)}$$

$$p_n^c = i\left(a_n^{c\dagger} - a_n^c\right)\sqrt{\frac{M\omega_n}{2}} = \frac{1}{2}i\sqrt{L\omega_n}\left(a_n^{c\dagger} - a_n^c\right)$$

with equivalent expressions for ϕ_n^s and p_n^s in terms of a_n^s and $a_n^{s\dagger}$.

Writing P in terms of a's & $a^\dagger$s, we get:

$$P = \sum_{n=1}^{\infty}\frac{2\pi n}{L}\, i\left(-a_n^{c\dagger}a_n^s + a_n^{s\dagger}a_n^c\right)$$

From this expression, we see that P acting on $a_n^{\dagger c}|0\rangle$ will be proportional to $a_n^{\dagger s}|0\rangle$ and vice versa, so the momentum eigenstates will be linear combinations:

$$|\psi\rangle = \alpha\, a_n^{\dagger c}|0\rangle + \beta a_n^{\dagger s}|0\rangle$$

We have
$$\begin{aligned} & P|\psi\rangle \\ &= P\left(\alpha a_n^{\dagger c}|0\rangle + \beta a_n^{\dagger s}|0\rangle\right) \\ &= -\frac{2\pi n}{L}\left(-i\alpha a_n^{\dagger s}|0\rangle + i\beta a_n^{\dagger c}|0\rangle\right) \end{aligned}$$

Demanding $P|\psi\rangle = p|\psi\rangle$, we have:

$$+i\alpha\frac{2\pi n}{L} = p\beta$$
$$-i\beta\frac{2\pi n}{L} = p\alpha$$

The solutions are $\beta = +i\alpha$ with $p = \frac{2\pi n}{L}$ and $\beta = -i\alpha$ with $p = -\frac{2\pi n}{L}$. Thus, the momentum eigenstates are:

chosen for proper normalization

$$\left|p = \frac{2\pi n}{L}\right\rangle = \frac{1}{\sqrt{2}}\left(a_n^{\dagger c} + i a_n^{\dagger s}\right)|0\rangle \equiv a_n^{\dagger}|0\rangle \quad n \geq 1$$
$$\left|p = -\frac{2\pi n}{L}\right\rangle = \frac{1}{\sqrt{2}}\left(a_n^{\dagger c} - i a_n^{\dagger s}\right)|0\rangle \equiv a_{-n}^{\dagger}|0\rangle \quad n \geq 1$$

We also have the special case $n=0$:

$$|p=0\rangle = a_0^{\dagger c}|0\rangle = a_0^{\dagger}|0\rangle$$

We can now go back and write the field $\phi(x)$ in terms of these.

We have:

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n^c \cos\left(\frac{2\pi n x}{L}\right) + \sum_{n=1}^{\infty} \phi_n^s \sin\left(\frac{2\pi n x}{L}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{L\omega_n}} \left(a_n^c + a_n^{c\dagger}\right) \cos\left(\frac{2\pi n x}{L}\right) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{L\omega_n}} \left(a_n^s + a_n^{s\dagger}\right) \sin\left(\frac{2\pi n x}{L}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{L\omega_n}} \left(a_n^c + a_n^{c\dagger}\right) \frac{e^{\frac{2\pi i n x}{L}} + e^{-\frac{2\pi i n x}{L}}}{2}$$

$$+ \frac{1}{\sqrt{L\omega_n}} \left(a_n^s + a_n^{s\dagger}\right) \frac{e^{\frac{2\pi i n x}{L}} - e^{-\frac{2\pi i n x}{L}}}{2i}$$

$$= \frac{1}{2\sqrt{L\omega_0}} \left(a_n^c + a_n^{c\dagger}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2\sqrt{L\omega_n}} e^{\frac{2\pi i n x}{L}} \left(\overbrace{a_n^{c\dagger} - i a_n^{s\dagger}}^{a_{-n}^{\dagger}} + \overbrace{a_n^c - i a_n^s}^{a_{+n}} \right)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2\sqrt{L\omega_n}} e^{-\frac{2\pi i n x}{L}} \left(\underbrace{a_n^{c\dagger} + i a_n^{s\dagger}}_{a_{+n}^{\dagger}} + \underbrace{a_n^c + i a_n^s}_{a_{-n}} \right)$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{L\omega_n}} \left\{ e^{-\frac{2\pi i n x}{L}} \left(a_n^{\dagger}\right) + \left(a_n\right) e^{+\frac{2\pi i n x}{L}} \right\}$$

$$\Rightarrow \quad \phi(x) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{L\omega_n}} \left\{ a_n^{\dagger} e^{-\frac{2\pi i n x}{L}} + a_n e^{+\frac{2\pi i n x}{L}} \right\}$$

or

$$\boxed{\phi(x) = \frac{1}{2} \sum_{p=\frac{2\pi n}{L}} \frac{1}{\sqrt{L\omega_n}} \left\{ a_p^{\dagger} e^{-ipx} + a_p e^{+ipx} \right\}}$$

c) We have classically:

$$\partial_t \phi(y) = \sum \dot{\phi}_n^c \cos\left(\frac{2n\pi x}{L}\right) + \dot{\phi}_n^s \sin\left(\frac{2n\pi x}{L}\right)$$

In terms of the conjugate momenta $p_n^c = \frac{L}{2}\dot{\phi}_n^c$, we get

$$\pi(y) = \frac{2}{L}\sum p_n^c \cos\left(\frac{2n\pi x}{L}\right) + p_n^s \sin\left(\frac{2n\pi x}{L}\right)$$

Quantum mechanically:

$$[\phi_n^c, p_n^c] = i\hbar$$
$$[\phi_n^s, p_n^s] = i\hbar$$

(all other commutators vanish)

So: $[\phi(x), \pi(y)]$

$$= \left[\sum_n \phi_n^c \cos\left(\frac{2n\pi x}{L}\right) + \phi_n^s \sin\left(\frac{2n\pi x}{L}\right), \frac{2}{L}\sum_m p_m^c \cos\left(\frac{2m\pi y}{L}\right) + p_m^s \sin\left(\frac{2m\pi y}{L}\right)\right]$$

$$= \frac{2i}{L}\sum_{n=0}^{\infty}\left(\cos\left(\frac{2n\pi x}{L}\right)\cos\left(\frac{2n\pi y}{L}\right) + \sin\left(\frac{2n\pi x}{L}\right)\sin\left(\frac{2n\pi y}{L}\right)\right) \quad (\text{setting } \hbar = 1)$$

$$= \frac{2i}{L}\sum_{n=0}^{\infty}\left(2e^{(inx-iny)\cdot 2\pi} + 2e^{(iny-inx)2\pi}\right)\cdot\frac{1}{4}$$

$$= \frac{i}{L}\sum_{n=-\infty}^{\infty} e^{in(x-y)\cdot 2\pi}$$

$$= i\,\delta(x-y)$$

to show that this is a delta fn, we can integrate against $f(x) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x/L}$ and show the result is $f(y)$.