Problem Set 4

Problem 1

a) Make sure you have read the notes on special relativity

b) Write out all the terms in the expression $\partial_\mu \phi \partial^\mu \phi$.

c) The scalar and vector potentials in electromagnetism form a four-vector field

$$A^\mu(t, \vec{x}) = \begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

under Lorentz transformations. Starting with the potential

$$A^\mu(t, \vec{x}) = \begin{pmatrix} Ey/c \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

(i.e. a constant electric field), what is the potential after a boost by velocity v in the x direction (last equation on page 1 of the notes)?

Problem 2

a) A spin $\frac{1}{2}$ particle at rest is in the state $|j = \frac{1}{2} m = \frac{1}{2}\rangle$. If we make an infinitesimal rotation about the x axis on this state by an angle $\delta\theta$, what is the change in the state?

b) For the same initial state, suppose we make a 45 degree rotation about the y axis. Write a formal expression for the new state (you can evaluate it explicitly if you want, but you will get full credit as long as your expression is correct).

Problem 3

Consider a one-dimensional field theory (choosing units so $c = 1$ and $\hbar = 1$) with action

$$S = \int dt \int_0^L dx \{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \} .$$

but with *periodic* boundary conditions $\phi(x + L) = \phi(x)$.

a) Since the theory is now translation-invariant in the x-direction, momentum is conserved, and we should be able to find a basis of energy eigenstates that are also momentum eigenstates. Determine the energies and momenta of the single-particle states for this theory and write expressions for these in terms of some creation and annihilation operators associated with the field modes.

b) Write an expression for the quantum field $\phi(x)$ in terms of creation and annihilation operators $a_p^\dagger$ and a_p that create and destroy particles with momentum p. These might be linear combinations of the creation and annihilation operators that you introduce for part a).

c) Show that

$$[\phi(x), \pi(y)] = i\delta(x-y)$$

where $\pi(y)$ is the quantum operator associated with $\partial_t\phi(y)$. You may find it useful to use

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} e^{2in\pi x/L} = \delta(x) \; .$$

d) Describe what happens to the allowed particle energies and momenta as we take $L \to \infty$.