② a) For $S = \int dt \int dx \left\{ \frac{1}{2}\dot\phi^2 (1+\lambda\phi^2) + \alpha \phi^2 (\phi')^2 \right\}$ we have under a small variation in ϕ

$$\delta S = \int dt \int dx \{ \dot\phi \delta\dot\phi (1+\lambda\phi^2) + \lambda \dot\phi^2 \phi \delta\phi + 2\alpha\phi\delta\phi(\phi')^2 + 2\alpha\phi^2\phi'\delta\phi' \}$$

$$= \int dt \int dx \{ \delta\phi \{ -\ddot\phi(1+\lambda\phi^2) - \lambda\phi\dot\phi^2 - 2\alpha\phi(\phi')^2 - 2\alpha\phi^2\phi''$$

where we have integrated by parts and assumed $\delta\phi = 0$ at the initial & final time, and that $\phi \to 0$ as $|x| \to \infty$.

For this to vanish with any $\delta\phi$, we have:

$$\boxed{\ddot\phi(1+\lambda\phi^2) + \lambda\phi\dot\phi^2 + 2\alpha\phi(\phi')^2 + 2\alpha\phi^2\phi'' = 0}$$

b) Time translation symmetry acts as:

$$\phi(x,t) \to \phi(x,t+\varepsilon) \approx \phi(x,t) + \varepsilon\dot\phi(x,t) + \mathcal{O}(\varepsilon^2)$$

so $\delta\phi = \varepsilon\,\dot\phi(x,t)$. To derive the energy density & current, we consider $\delta\phi = \varepsilon(x,t)\,\dot\phi(x,t)$. We find

$$\delta S = \int dt\,dx \{ \dot\phi (1+\lambda\phi^2)(\dot\varepsilon\dot\phi + \varepsilon\ddot\phi) + \lambda\dot\phi^3\phi\varepsilon + 2\alpha\phi(\phi')^2(\varepsilon\dot\phi) + 2\alpha\phi^2\phi'(\varepsilon'\dot\phi + \varepsilon\dot\phi') \}$$

$$= \int dt\,dx \{ \dot\varepsilon(\dot\phi^2(1+\lambda\phi^2)) + \varepsilon'(2\alpha\phi^2\dot\phi(\phi')) + \varepsilon(\dot\phi\ddot\phi(1+\lambda\phi^2) + \lambda\phi\dot\phi^3 + 2\alpha\phi\dot\phi(\phi')^2 + 2\alpha\phi^2\phi'\dot\phi')$$

write this as

$$\frac{d}{dt}\left(\frac{1}{2}\dot\phi^2 + \frac{\lambda}{2}\phi^2\dot\phi^2\right) + \frac{d}{dt}\left(\alpha\phi^2(\phi')^2\right)$$

integrate by parts.

So: $$\delta S = \int dt\,dx\ \dot{\varepsilon}\left(\dot{\phi}^2(1+\lambda\phi^2)\right) + \varepsilon'\left(2\alpha\phi^2\dot{\phi}\phi'\right) + \varepsilon\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2(1+\lambda\phi^2) + \alpha\phi^2(\phi')^2\right)$$

integrate by parts.

$$= \int dt\ \dot{\varepsilon}\left(\frac{1}{2}\dot{\phi}^2(1+\lambda\phi^2) - \alpha\phi^2(\phi')^2\right) + \varepsilon'\left(2\alpha\phi^2\dot{\phi}\phi'\right)$$

Thus, the energy density is

$$\boxed{\rho = \frac{1}{2}\dot{\phi}^2(1+\lambda\phi^2) - \alpha\phi^2(\phi')^2}$$

The energy current is

$$\boxed{S = 2\alpha\phi^2\dot{\phi}\phi'.}$$

Spatial translations act as:

$$\tilde{\phi}(x,t) = \phi(x+a,t)$$

So $$\delta\phi = \left(\phi(x+\varepsilon,t) - \phi(x,t)\right]_{\text{order } \varepsilon} = \varepsilon\phi'$$

To derive the momentum density & current, we consider

$$\delta\phi = \varepsilon(x,t)\,\phi'$$

Then: $$\delta S = \int dx\,dt\left\{\dot{\phi}\left((1+\lambda\phi^2)(\dot{\varepsilon}\phi' + \varepsilon\dot{\phi}')\right) + \lambda\dot{\phi}^2\phi\,\varepsilon\phi' + 2\alpha\phi(\phi')^2\varepsilon\phi' + 2\alpha\phi^2\phi'(\varepsilon'\phi' + \varepsilon\phi'')\right\}$$

$$= \int dx\,dt\left\{\dot{\varepsilon}\left(\dot{\phi}\phi'(1+\lambda\phi^2)\right) + \varepsilon'\left(2\alpha\phi^2(\phi')^2\right) + \varepsilon\left(\dot{\phi}\dot{\phi}'(1+\lambda\phi^2) + \lambda\dot{\phi}^2\phi\phi' + 2\alpha\phi(\phi')^3 + 2\alpha\phi^2\phi'\phi''\right)\right.$$

↑

$$\frac{d}{dx}\left(\frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{2}\phi^2\dot{\phi}^2 + \alpha\phi^2(\phi')^2\right)$$

So: $\delta S = \int dx\,dt \left\{ \dot{\varepsilon}\left(\dot{\phi}\phi'(1+\lambda\phi^2)\right) + \varepsilon'\left(2\alpha\phi^2(\phi')^2\right) + \varepsilon \frac{d}{dx}\left(\frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{2}\phi^2\dot{\phi}^2 + \alpha\phi^2(\phi')^2\right)\right\}$ ↓ int. by parts

$$= \int dx\,dt \; \dot{\varepsilon}\left(\dot{\phi}\phi'(1+\lambda\phi^2)\right) + \varepsilon'\left(\alpha\phi^2(\phi')^2 - \frac{1}{2}\dot{\phi}^2 - \frac{\lambda}{2}\phi^2\dot{\phi}^2\right)$$

Thus, the momentum density is

$$\boxed{p = \dot{\phi}\phi'(1+\lambda\phi^2)}$$

and the momentum current is

$$\boxed{J_p = \alpha\phi^2(\phi')^2 - \frac{1}{2}\dot{\phi}^2 - \frac{\lambda}{2}\phi^2\dot{\phi}^2}$$

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(3) a) For $S = \int dt \int dx \left\{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\phi')^2\right\}$ and $\Phi_{(x,t)} = \Lambda^n \phi\left(\frac{x}{\Lambda}, \frac{t}{\Lambda}\right)$

we find that:

$$S[\Phi] = \int dt \int dx \left\{\frac{1}{2}\left(\Lambda^n \cdot \frac{1}{\Lambda}\dot{\phi}\left(\frac{x}{\Lambda},\frac{t}{\Lambda}\right)\right)^2 - \frac{1}{2}\left(\Lambda^n \cdot \frac{1}{\Lambda}\phi'\left(\frac{x}{\Lambda},\frac{t}{\Lambda}\right)\right)^2\right\}$$

$$= \Lambda^{2n}\int \frac{dt}{\Lambda}\int\frac{dx}{\Lambda}\left\{\frac{1}{2}\left(\dot{\phi}\left(\frac{x}{\Lambda},\frac{t}{\Lambda}\right)\right)^2 - \frac{1}{2}\left(\phi'\left(\frac{x}{\Lambda},\frac{t}{\Lambda}\right)\right)^2\right\}$$

$$= \Lambda^{2n}\int d\tilde{t}\,d\tilde{x}\left\{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\phi')^2\right\}$$

This equals $S[\phi]$ if we take $n=0$.

The symmetry transform is trivial if $\Lambda = 1$, so to get the infinitesimal version, we take $\Lambda = 1+\varepsilon$. Then

$$\tilde{\Phi}(x,t) = \phi\left(\frac{x}{1+\varepsilon}, \frac{t}{1+\varepsilon}\right) = \phi(x-\varepsilon x+\ldots, t-\varepsilon t+\ldots)$$

$$= \phi(x,t) - \varepsilon x\,\phi'(x,t) - \varepsilon t\,\dot{\phi}(x,t) + \mathcal{O}(\varepsilon^2)$$

So: $\delta\phi = -\varepsilon x\,\phi'(x,t) - \varepsilon t\,\dot{\phi}(x,t)$

Plugging in $\delta\phi = -\varepsilon(x,t)\cdot x\cdot\phi' - \varepsilon(x,t)\cdot t\cdot\dot{\phi}$, we get

$$\delta S = \int dt dx \{ \dot{\phi}(-\dot{\varepsilon}x\phi' - \varepsilon x\dot{\phi}' - \dot{\varepsilon}t\dot{\phi} - \varepsilon\dot{\phi} - \varepsilon t\ddot{\phi})$$
$$-\phi'(-\varepsilon' x\phi' - \varepsilon\phi' - \varepsilon x\phi'' - \varepsilon' t\dot{\phi} - \varepsilon t\dot{\phi}')$$
$$= \int dt dx \quad \dot{\varepsilon}(-\dot{\phi}x\phi' - t\dot{\phi}^2) + \varepsilon'((\phi')^2 x + t\dot{\phi}\phi')$$
$$+\varepsilon(-x\dot{\phi}\dot{\phi}' - \dot{\phi}^2 - t\dot{\phi}\ddot{\phi} + (\phi')^2 + x\phi'\phi'' + t\phi'\dot{\phi}')$$

second line is this: $\varepsilon\left(\frac{d}{dt}\left(-\frac{1}{2}t\dot{\phi}^2\right) + \frac{d}{dx}\left(-\frac{1}{2}x\dot{\phi}^2\right) + \frac{d}{dt}\left(\frac{1}{2}t(\phi')^2\right) + \frac{d}{dx}\left(\frac{1}{2}x(\phi')^2\right)\right)$

$$= \int dt dx \quad \dot{\varepsilon}(-\dot{\phi}x\phi' - \tfrac{1}{2}t\dot{\phi}^2 - \tfrac{1}{2}t(\phi')^2)$$
$$+\varepsilon'\left(\tfrac{1}{2}x(\phi')^2 + t\dot{\phi}\phi' + \tfrac{1}{2}x\dot{\phi}^2\right)$$

Thus: the density is $J^t = -\frac{1}{2}t(\dot{\phi}^2 + (\phi')^2) - \dot{\phi}x\phi'$

the current is $J^x = \frac{1}{2}x(\dot{\phi}^2 + (\phi')^2) + t\dot{\phi}\phi'$

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4) The infinitesimal version of the symmetry is

$$\delta\phi_x = -\varepsilon\phi_y$$
$$\delta\phi_y = \varepsilon\phi_x$$

Promoting $\varepsilon \to \varepsilon(t)$, we find

$$\delta S = \int dt \int dx \{ \rho\dot{\phi}_x(-\varepsilon\dot{\phi}_y - \dot{\varepsilon}\phi_y)$$
$$+\rho\dot{\phi}_y(+\varepsilon\dot{\phi}_x + \dot{\varepsilon}\phi_x)$$
$$-\tau\phi_x'(-\varepsilon\phi_y')$$
$$-\tau\phi_y'(\varepsilon\phi_x')\}$$
$$= \int dt \int dx \{ \dot{\varepsilon}(\rho\dot{\phi}_y\phi_x - \rho\dot{\phi}_x\phi_y)\}$$

The conserved charge is then

$$Q = \int_0^L dx\,(\rho\,\dot{\phi}_y\phi_x - \dot{\phi}_x\phi_y)$$

In terms of the modes, we find

$$Q = \sum_n \tfrac{1}{2}\rho L \left(\dot\phi^y_n \phi^x_n - \dot\phi^x_n \phi^y_n \right)$$

(where we have expanded $\phi_x = \sum \phi^x_n \sin\left(\frac{n\pi x}{L}\right)$, $\phi_y = \sum \phi^y_n \sin\left(\frac{n\pi x}{L}\right)$)

b) The momenta conjugate to ϕ^x_n and ϕ^y_n are

$$p^{x,y}_n = \frac{\rho L}{2} \dot\phi^{x,y}_n$$

Thus:

$$Q = \sum_n \left(p^y_n \phi^x_n - p^x_n \phi^y_n \right)$$

Defining creation & annihilation operators $a^{\dagger x,y}_n$ and $a^{x,y}_n$ by

$$a^{\dagger x,y}_n = \frac{1}{\sqrt{\hbar \rho L \omega_n}} \left(-i p^{x,y}_n + \frac{\rho L}{2} \omega_n \phi^{x,y}_n \right)$$

we find:

$$Q = \sum_n i\hbar \left(a^x_n a^{y\dagger}_n - a^y_n a^{x\dagger}_n \right)$$

c) The single particle states of definite energy are linear combinations

$$|\psi\rangle = \alpha\, a^{\dagger x}_n |0\rangle + \beta\, a^{\dagger y}_n |0\rangle$$

Demanding that $Q|\psi\rangle = q|\psi\rangle$ gives

$$|\psi^+_n\rangle = (a^{\dagger x}_n + i a^{\dagger y}_n)|0\rangle \quad \text{with } q = \hbar$$

$$|\psi^-_n\rangle = (a^{\dagger x}_n - i a^{\dagger y}_n)|0\rangle \quad \text{with } q = -\hbar$$

Thus the particles may be divided into sets with charge $\pm\hbar$