Homework 2 solutions

1a) To evaluate $\langle 0 | \phi^2(\frac{L}{2}) | 0 \rangle$, note that

$$\phi\left(\tfrac{L}{2}\right) = \sum_n \phi_n \sin\left(\tfrac{n\pi}{2}\right)$$
$$= \sum_k \phi_{2k+1} (-1)^k$$

$$\therefore \langle 0 | \phi^2\left(\tfrac{L}{2}\right) | 0 \rangle = \langle 0 | \sum_{k,\ell} \phi_{2k+1} (-1)^k \phi_{2\ell+1} (-1)^\ell | 0 \rangle$$

$$= \langle 0 | \sum_\ell \phi_{2\ell+1}^2 | 0 \rangle$$

(all other terms have the form $\langle 0 | \phi_n \phi_m | 0 \rangle$ for two independent modes, so give $\langle 0 | \phi_n | 0 \rangle \langle 0 | \phi_m | 0 \rangle = 0$

$$= \sum_\ell \langle 0 | \frac{2}{M\omega_{2\ell+1}^2} U_{2\ell+1} | 0 \rangle$$

potential energy term for $2\ell+1$'st mode

$$M \equiv \tfrac{L}{2}\rho$$
$$\omega_{2\ell+1} = \sqrt{\tfrac{\tau}{\rho}} \left(\tfrac{\pi}{L}(2\ell+1)\right)$$

$$= \sum_\ell \frac{2}{M\omega_{2\ell+1}^2} \cdot \frac{1}{2} \mathcal{E}_{2\ell+1}$$

(using the Virial theorem $\langle U \rangle = \langle T \rangle = \frac{1}{2} \langle E \rangle$

~~[illegible]~~
alternatively, just expand $\phi_{2\ell+1}$ in terms of $a_{2\ell+1}, a_{2\ell+1}^\dagger$ and calculate directly

$$= \sum_\ell \frac{\hbar}{2M\omega_{2\ell+1}}$$

$$= \frac{\hbar}{\sqrt{\tau\rho}} \frac{1}{\pi} \sum_{\ell=1}^{\infty} \frac{1}{2\ell+1} \rightarrow \infty$$

Thus, the result diverges!

Assuming $\lambda_{min} = \dfrac{2L}{n_{max}} = 1\,\text{Å}$, we have

$$\langle \phi^2(\tfrac{L}{2}) \rangle \approx \frac{\hbar}{\sqrt{\tau\rho}} \cdot \frac{1}{\pi} \sum_{l=0}^{\lfloor \frac{L}{1\text{Å}} \rfloor} \frac{1}{2l+1}$$

$$\approx \frac{\hbar}{\pi\sqrt{\tau\rho}} \left\{ \ln\left(\sqrt[2]{\frac{L}{1\,\text{Å}}}\right) + \frac{\gamma}{2} \right\}$$

I used $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \approx \ln(n) + \gamma$

γ = Euler's constant

$\frac{1}{2}\ln\left(\frac{L}{1\,\text{Å}}\right)$ would be fine

For $\tau = 1\,N$, $\rho = 10\,g/m$, $L = 1\,m$, we find

$$\langle \phi^2(\tfrac{L}{2}) \rangle \approx 10^{-33}\,m^2$$

so $\sqrt{\phi^2(\frac{L}{2})} \approx 3 \times 10^{-17}\,m$, smaller than the size of a nucleus!