PROBLEM SET 1 SOLUTIONS

We can use the same change of variables as on the worksheet,

$$\phi(x,t) = \sum_n \phi_n(t) \sin\left(\frac{n\pi x}{L}\right).$$

The equations of motion become:

$$\rho \sum_n \ddot{\phi}_n \sin\left(\frac{n\pi x}{L}\right) = \tau \sum_n \phi_n(t) \left\{ -\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \right\}$$
$$- \mu \sum_n \phi_n(t) \left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$$\Rightarrow \quad \sum_n \sin\left(\frac{n\pi x}{L}\right) \left[\rho \ddot{\phi}_n + \left\{ \tau \left(\frac{n\pi}{L}\right)^2 + \mu \right\} \phi_n \right] = 0$$

The Fourier decomposition of 0 has all terms equal to 0, so we find

$$\ddot{\phi}_n = -\frac{1}{\rho} \left\{ \tau \left(\frac{n\pi}{L}\right)^2 + \mu \right\} \phi_n .$$

The variable ϕ_n has the same e.o.m. as the displacement of a harmonic oscillator with freq. $\omega_n = \sqrt{\frac{1}{\rho}\left(\tau\left(\frac{n\pi}{L}\right)^2 + \mu\right)}$.

For the energy, the change of variables yields

$$E = \int_0^L dx \left\{ \frac{1}{2}\rho \left(\sum_n \dot{\phi}_n \sin\left(\frac{n\pi x}{L}\right)\right)\left(\sum_m \dot{\phi}_m \sin\left(\frac{m\pi x}{L}\right)\right) \right.$$
$$+ \frac{1}{2}\tau \left(\sum_n \phi_n \left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right)\left(\sum_m \phi_m \left(\frac{m\pi}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\right)$$
$$\left. + \frac{1}{2}\mu \left(\sum_n \phi_n \sin\left(\frac{n\pi x}{L}\right)\right)\left(\sum_m \phi_m \sin\left(\frac{m\pi x}{L}\right)\right) \right\}$$

$$= \sum_n \frac{\rho L}{4} \dot{\phi}_n^2 + \left\{\frac{\tau L}{4}\left(\frac{n\pi}{L}\right)^2 + \frac{\mu L}{4}\right\}\phi_n^2$$

where we have used

$$\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \frac{L}{\pi}\int_0^\pi dy \sin(ny)\sin(my)$$
$$= \frac{L}{2} \times \delta_{n,m}$$

and $\int_0^L dx \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \frac{L}{2}\delta_{n,m}$.

This matches with the energy for a bunch of harmonic oscillators

$$E = \sum_n \frac{1}{2}m_n \dot{x}_n^2 + \frac{1}{2}m_n \omega_n^2 x_n^2$$

if we define $m_n = \frac{\rho L}{2}$

$$\omega_n = \sqrt{\frac{\tau}{\rho}\left(\frac{n\pi}{L}\right)^2 + \frac{\mu}{\rho}}.$$

The field theory is mathematically equivalent to a system of harmonic oscillators at the classical level, so the quantum spectrum for the two systems is the same. For the subsystem described by ϕ_n, the allowed quantum energies are:

$$E - E_0 = \hbar \omega_n N_n$$
$$= \hbar N_n \sqrt{\frac{\tau}{\rho}\left(\frac{n\pi}{L}\right)^2 + \frac{\mu}{\rho}}$$

The allowed energies for the full system are

$$\boxed{E - E_0 = \sum_n \hbar N_n \sqrt{\frac{\tau}{\rho}\left(\frac{n\pi}{L}\right)^2 + \frac{\mu}{\rho}}}$$

b) For particles of mass M, the energies are given by

$$E = \sqrt{M^2 c^4 + p^2 c^2}$$

the allowed momenta are such that the de Broglie wave length $\lambda = \frac{h}{p}$ is $2L, L, \frac{2L}{3}, \frac{L}{2}, \ldots$ (we want an integer or half integer number of wavelengths to fit in the box. Thus, the allowed energies for a single particle are

$$E_n = \sqrt{M^2 c^4 + c^2 \cdot \frac{h^2}{\lambda_n^2}} = \sqrt{M^2 c^4 + \frac{c^2 h^2 n^2}{(2L)^2}}$$

We can have any number of particles at each wave length (* if they are bosons *) so the total energy can be:

$$\boxed{E = \sum_n \mathcal{N}_n \cdot \sqrt{M^2c^4 + \frac{c^2\hbar^2 n^2}{(2L)^2}}} \quad (*)$$

c) If we take $\frac{\tau}{\rho} = c^2$, the field theory energies are:

$$E - E_0 = \sum_n N_n \sqrt{\frac{\hbar^2 \mu}{\rho} + \frac{c^2\hbar^2 n^2}{(2L)^2}}$$

This matches exatly with (*) if we take

$$\boxed{\mu = \frac{\rho M^2 c^4}{\hbar^2}}$$