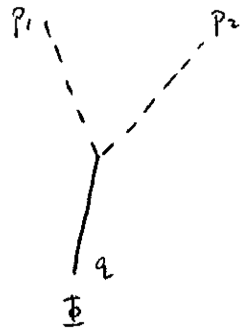
① The amplitude is



$$i\mathcal{M} = -i\mu \cdot \underset{\uparrow}{2}$$

\# ways to connect diagram

The decay rate is then:

$$\Gamma = \frac{1}{2M}\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_{p_1}}\int \frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_{p_2}}\; \underbrace{4\mu^2}_{|\mathcal{M}|^2}\,(2\pi)^4\,\delta^4(p_1+p_2-q)$$

In the center of mass frame (which we assume) $E_{p_1} = E_{p_2}$

$$\Gamma = \frac{1}{2M}\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{4E_{p_1}^2}\cdot 4\mu^2\,(2\pi)\,\delta(2E_{p_1}-M)$$

$$= \frac{\mu^2}{2M}\int d\Omega \int_0^\infty \frac{p_1^2\,dp_1}{\left(\sqrt{p_1^2+m^2}\right)^2}\,(2\pi)^{-2}\cdot\delta\left(2\sqrt{p_1^2+m^2}-M\right)$$

$$= \frac{\mu^2}{2M}\int d\Omega\;\frac{p_1^2}{p_1^2+m^2}\,(2\pi)^{-2}\cdot\frac{\sqrt{p_1^2+m^2}}{2p_1}$$

$$= \frac{\mu^2}{8\pi^2 M^2}\cdot p_1 \int d\Omega$$

We need only integrate over the half sphere to get all the events, since the outgoing particles are identical.

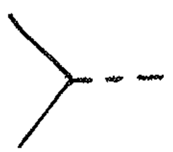
Thus, $$\Gamma = \frac{\mu^2}{4\pi M^2}\cdot\sqrt{\frac{M^2}{4}-m^2}$$

$$\boxed{\Gamma = \frac{\mu^2}{8\pi M}\sqrt{1-\left(\frac{2m}{M}\right)^2}}$$

② We would like to compute the cross section for the process $\phi\phi \to \phi\phi$, in a theory with ϕ particles of mass m and scalar Φ particles of mass M, with an interaction Hamiltonian density

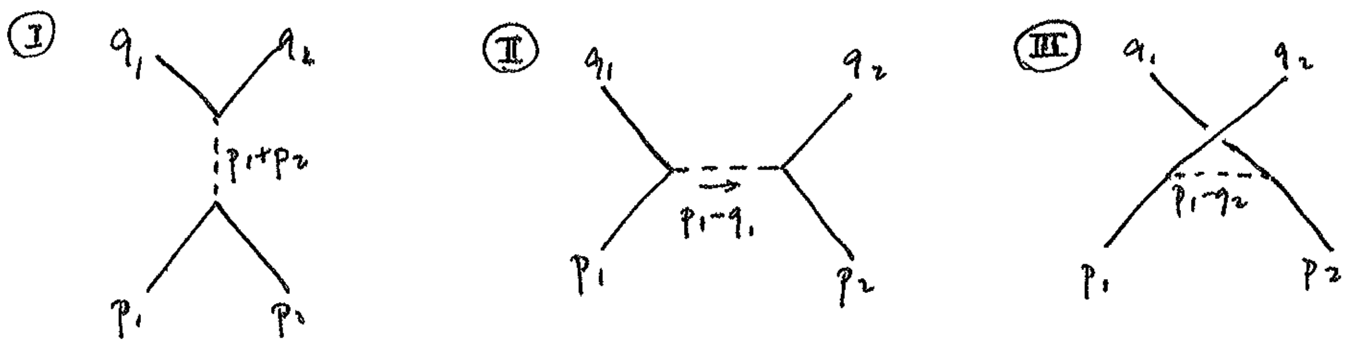
$$\mathcal{H}_I = \mu \Phi \phi\phi$$

The Feynman rule for this vertex is



$-i\mu$

where we represent ϕ and Φ particles with solid and dashed lines respectively. There are 3 diagrams contributing:



Using the Feynman rules, these give:

$$i\mathcal{M}_I = \frac{1}{2}(-i\mu)^2 \cdot 4 \cdot 2 \cdot \Delta_F^M(p_1+p_2)$$

$$= \frac{-4\mu^2}{(p_1+p_2)^2 - M^2}$$

$$i\mathcal{M}_{II} = \frac{-4\mu^2}{(p_1-q_1)^2 - M^2}$$

$$i\mathcal{M}_{III} = \frac{-4\mu^2}{(p_1-q_2)^2 - M^2}$$

Overall, we have

$$
\begin{aligned}
|M|^2 &= |M_{\mathrm{I}} + M_{\mathrm{II}} + M_{\mathrm{III}}|^2 \\
&= 16\mu^4 \left(\frac{1}{(p_1+p_2)^2 - M^2} + \frac{1}{(p_1-q_1)^2 - M^2} + \frac{1}{(p_1-q_2)^2 - M^2} \right)^2
\end{aligned}
$$

We now plug this into the formula for the cross section in the center of mass frame, where $p_1 = (E, \vec{p})$, $p_2 = (E, -\vec{p})$, $q_1 = (E, \vec{q})$, $q_2 = (E, -\vec{q})$

This gives:

$$
\begin{aligned}
|M|^2 &= 16\mu^4 \left(\frac{1}{4E^2 - M^2} - \frac{1}{(\vec{p}-\vec{q})^2 + M^2} - \frac{1}{(\vec{p}+\vec{q})^2 + M^2} \right)^2 \\
&= 16\mu^4 \left(\frac{1}{4E^2 - M^2} - \frac{1}{2E^2 + M^2 - 2m^2 - 2(E^2-m^2)\cos\theta} - \frac{1}{2E^2 + M^2 - 2m^2 + 2(E^2-m^2)\cos\theta} \right)^2
\end{aligned}
$$

The rest of the calculation is the same as for the ϕ^4 theory, so we get the same result with the replacement $\lambda^2 \to |M|^2$, i.e.

$$
\frac{d\sigma}{d\Omega} = \frac{\mu^4}{16\pi^2 E^2} \underbrace{\left(\frac{1}{4E^2 - M^2} - \frac{4E^2 + 2M^2 - 4m^2}{(2E^2 + M^2 - 2m^2)^2 - 4(E^2 - m^2)^2 \cos^2\theta} \right)^2}_{f^2(\cos\theta)}
$$

b) The total cross section is the integral of $\frac{d\sigma}{d\Omega}$ over the half sphere, so for $M=4m$, we find:

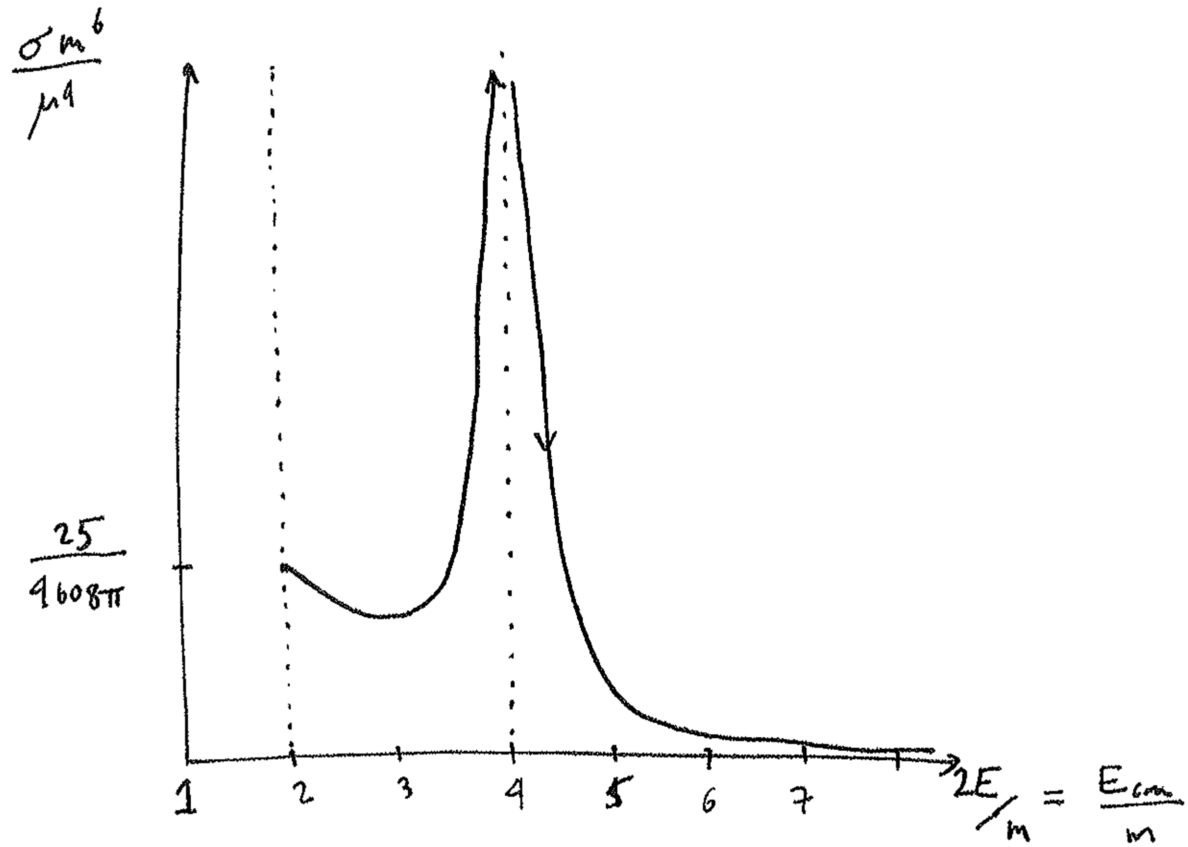
$$\sigma = \frac{\mu^4}{16\pi^2 E^2}\,\pi \int_0^\pi \sin\theta\, d\theta\, f^2(\cos\theta)$$

$$= \frac{\mu^4}{16\pi E^2}\int_{-1}^{1} dx\, f^2(x)$$

$$= \frac{\mu^4}{16\pi E^2}\left\{\frac{E^4 - 6E^2m^2 + 22m^4}{m^2\cdot 16\cdot (E^2-4m^2)^2 (E^2+3m^2)} - \frac{11}{2}\,\frac{\ln\left((E^2+3m^2)/4m^2\right)}{(E^2-4m^2)(E^2+7m^2)(E^2-m^2)}\right\}$$

(used Maple)..

Now, $\frac{\sigma}{\mu^4}\cdot m^6$ is a function only of E/m, so we can plot this, giving:



Note $E_{cm} \geq 2m$

c) For $E \ll M$ and $m \ll M$, we find

$$\frac{d\sigma}{d\Omega} = \frac{9}{16\pi^2}\frac{\mu^4}{M^4 E^2},$$ independent of θ. Thus, at

energies much below the Φ mass, we could accurately reproduce the results of experiments using an interaction

$$\mathcal{H}_I = \frac{\mu^2}{2M^2}\phi^4$$

Only by doing experiments at energies close to M could we discover that this ϕ^4 interaction doesn't correctly describe the physics. The peak in the cross section at $E_{cm} = 4m$ indicates the existence of a new particle with mass $4m$.