Vector Fields

The most general Poincaré-invariant local quadratic action for a vector field with no more than first derivatives on the fields (ensuring that classical evolution is determined based on the fields and first time derivatives) is

$$S = \int d^{d+1}x\{a\partial_\mu\phi^\nu\partial^\mu\phi_\nu + b\partial_\mu\phi^\nu\partial_\nu\phi^\mu + c\phi_\mu\phi^\mu\}$$

There appear to be three free parameters, but it turns out that we can reduce to only one parameter by physical considerations. First, we notice that the term $\phi_\mu\phi^\mu = (\phi^0)^2 - (\phi_i)^2$ implies that either the time component ϕ_0 or the spatial components ϕ_i will have negative quadratic terms in the potential energy. Similarly, including the term $\partial_\mu\phi^\nu\partial^\mu\phi_\nu$ will give either $(\dot{\phi}^t)^2$ or $(\dot{\phi}^i)^2$ terms in the kinetic energy with the wrong sign. In either case, the classical energy is unbounded below if arbitrary field configurations are allowed. To make sense of this, consider the following:

Q: What is the equation of motion for a single particle system whose action is simply $S = \int dt - \frac{1}{2}x^2$ with no kinetic energy term?

Answer: In this case, extremizing the action requires that $x = 0$.

In this simple example, we see that the "equation of motion" is actually just a constraint on the physical variables. Rather than allowing arbitrary initial conditions for x and $\dot{x}$, the equation simply requires that $x = 0$, effectively eliminating one of the degrees of freedom.

In the vector field theory, our problem was that the kinetic and potential energies were unbounded below if we allow arbitrary field configurations. In this case, a constraint is exactly what we need. Focusing on the kinetic energy, we notice that by demanding $b = -a < 0$, the kinetic term $(\partial_t\phi_t)^2$ vanishes and all the remaining kinetic energy terms are positive. The equation of motion for ϕ^0 gives a constraint that allows us to solve for ϕ_0 in terms of the other fields. We will see that taking into account the constraint, the energy of the theory is bounded from below.

The theory we have ended up with can be reduced to one with a single free parameter by redefining the fields so that $a = 1/2$. If we also call $c = \frac{1}{2}m^2$, the final action

$$S = \int d^{d+1}x\{\frac{1}{2}\partial_\mu\phi^\nu\partial^\mu\phi_\nu - \frac{1}{2}\partial_\mu\phi^\nu\partial_\nu\phi^\mu + \frac{1}{2}m^2\phi_\mu\phi^\mu\}$$

It is standard to define $F_{\mu\nu} = \partial_\mu\phi_\nu - \partial_\nu\phi_\mu$, so that the action may be written compactly as

$$S = \int d^{d+1}x\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2\phi_\mu\phi^\mu\} \ .$$

As for the scalar field theory, we will find that the parameter m gives the mass of the particles that the quantum field theory describes. Remarkably, the equations of motion for the theory with $m = 0$ are exactly Maxwell's equations for electromagnetism expressed in terms of the scalar and vector potentials, if we interpret $\phi^\mu = (\phi, \vec{A})$, $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B_k$ (see homework 5 solutions for details). Thus, Maxwell's equations for the electromagnetic field can be understood to follow completely from the constraints of locality, Poincaré invariance, the superposition principle, the constraint that classical evolution is determined by the potentials and their first time derivatives, and the constraint that shifts $A_0 \to A_0 + \text{const}$ do not affect the action (a familiar property of scalar potential).

Quantization

Let's try to quantize this theory. According to the general rules, the first step is to determine the conjugate momenta for all of our fields

Q: What are the conjugate momenta to ϕ_0 and ϕ_i?

Answer: We find that

$$\begin{aligned} \pi_i &= \frac{\delta L}{\delta \dot{\phi}_i} = \partial_0 \phi_i - \partial_i \phi_0 \\ \pi_0 &= \frac{\delta L}{\delta \dot{\phi}_0} = 0 \ . \end{aligned} \tag{1}$$

Since there is no kinetic term for ϕ_0 in our action, we find that the canonically conjugate momentum for this field vanishes. In this case, it doesn't make sense to impose $[\phi_0(x), \pi_0(y)] = i\delta(x-y)$, so the standard quantization procedure won't work. On the other hand, we need to keep in mind that the absence of a kinetic term for ϕ_0 means that we are working with a constrained system. In particular, the equation of motion for ϕ_0 gives

$$\phi_0 = \frac{1}{m^2}\partial_i(\partial_i \phi_0 - \partial_0 \phi_i) = -\frac{1}{m^2}\partial_i \pi_i \tag{2}$$

so we can solve for ϕ_0 in terms of π_i. Thus, we can express all physical observables in terms of the spatial components ϕ_i and π_i, and pass to quantum mechanics by promoting these to operators satisfying

$$[\phi_i(x), \pi_j(y)] = i\delta_{ij}\delta^d(x-y) \ . \tag{3}$$

As an example, the Hamiltonian for the theory is given by

$$H = \int d^d x \dot{\phi}^i \pi^i - L \ .$$

We want to rewrite this in terms of ϕ_i and π_i, so we eliminate $\dot{\phi}_i$ using (1) and we eliminate ϕ_0 using (2). This gives

$$\begin{aligned} H &= \int d^d x \{(\pi^i - \frac{1}{m^2}\partial_i(\nabla\cdot\pi))\pi_i - \int d^d x \{\frac{1}{2}\pi_i\pi_i - \frac{1}{4}F_{ij}F_{ij} - \frac{1}{2}m^2\phi^i\phi^i + \frac{1}{2}m^2\frac{1}{m^4}(\nabla\cdot\pi)^2\} \\ &= \int d^d x \{\frac{1}{2}\pi_i\pi_i + \frac{1}{2m^2}(\nabla\cdot\pi)^2 + \frac{1}{4}(\partial_i\phi_j - \partial_j\phi_i)^2 + \frac{1}{2}m^2\phi^i\phi^i\} \end{aligned} \tag{4}$$

We see that all terms are positive, so the energy is bounded from below.

Creation and annihilation operators

The quantum theory is formally defined by giving the Hamiltonian (4) together with the commutation relations (3). As with scalar field theory, the physics can be extracted most easily if we can express everything in terms of a set of creation and annihilation operators. For scalar fields, the expression for the field in terms of creation and annihilation operators is the same as the expression for the general solution to the classical field equation, where the a_p is the coefficient of the negative-frequency plane-wave solution to the Klein-Gordon equation with wave number $\vec{p}$

$$\phi(x,t) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x})$$

and $a_p^\dagger$ is the coefficient of the positive frequency solution.

It turns out that this relation between the time-dependent field operator and the creation and annihilation operator works for all kinds of fields (as we can check).[1] Thus, to express the vector field in terms of creation and annihilation operators, our first step is to find the general solution to the classical equations of motion. These equations are

$$\partial_\mu(\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) + m^2 \phi^\nu = 0 \; .$$

Acting with ∂^ν (and summing over ν), we find

$$\partial_\nu \phi^\nu = 0 \; , \tag{5}$$

so that the original equation of motion simplifies to

$$(\partial^2 + m^2)\phi^\mu = 0 \; . \tag{6}$$

Thus, each component of the vector field satisfies the Klein-Gordon equation, but we must also satisfy the constraint (5) that relates the components. The plane-wave solution with wave-vector p can be written

$$\phi^\mu = \epsilon^\mu e^{ip\cdot x}$$

where the equation (6) implies

$$p^2 - m^2 = (p^0)^2 - \vec{p}^2 - m^2 = 0$$

and the equation (5) implies that

$$\epsilon^\mu p_\mu = 0 \; .$$

[1] In the case of complex fields, the coefficient of $e^{ip\cdot x}$ does not have to be the complex conjugate of the coefficient of $e^{-ip\cdot x}$. Accordingly, the expression for the field in terms of creation and annihilation operators is

$$\phi(x,t) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x})$$

where b_p and a_p are annihilation operators for separate types of particles. These particles have charge ± 1 for the charge associated with the classical symmetry $\phi \to e^{i\theta}\phi$.

Thus, the frequency must be $p^0 = \pm\sqrt{\vec{p}^2 + m^2}$. For each choice of $\vec{p}$, there will be three linearly independent choices for ϵ^μ, which we call ϵ_r^μ and can choose such that that $\epsilon_r^\mu(\epsilon_\mu)_s = \delta_{rs}$.[2] Then the general solution to the equations of motion is given by

$$\phi^\mu(x) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} (\alpha_{\vec{p}}^r \epsilon_r^\mu e^{-ip\cdot x} + (\alpha_{\vec{p}}^r)^* \epsilon_r^\mu e^{ip\cdot x})$$

where $\alpha_{\vec{p}}^r$ is an arbitrary complex number for each $\vec{p}$ and $r = 1, 2, 3$. The factors of $(2\pi)^d$ and $1/(2E_p)$ here are just inserted by convention.

Now, as for the scalar field theory, the expression for the time-dependent quantum field operator in terms of creation and annihilation operators is the same as the expression for the general solutions to the field equation, with the creation operator $a_{\vec{p}}^r$ taking the place of the coefficient $\alpha_{\vec{p}}^r$ of the negative frequency mode and the annihilation operator $(a_{\vec{p}}^r)^\dagger$ taking the place of the coefficient $(\alpha_{\vec{p}}^r)^*$ of the positive frequency mode. Thus, we have

$$\phi^\mu(x) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} (a_{\vec{p}}^r \epsilon_r^\mu e^{-ip\cdot x} + (a_{\vec{p}}^r)^\dagger \epsilon_r^\mu e^{ip\cdot x})$$

So far, this is just a guess, but we can verify that a and $a^\dagger$ have the desired commutation relations

$$[a_{\vec{p}}^r, (a_{\vec{q}}^s)^\dagger] = \delta_{rs}(2\pi)^d \delta^d(p - q)$$

by writing an expression for π_i in terms of as and $a^\dagger$s[3], and then checking that (**??**) is equivalent to (3). We can also verify that

$$[H, (a_{\vec{p}}^s)^\dagger] = E_p (a_{\vec{p}}^s)^\dagger \ ,$$

so all the usual properties for creation and annihilation operators are satisfied.

Physics

By writing the energy and momentum in terms of creation and annihilation operators, we can verify that each of the operators $(a_{\vec{q}}^s)^\dagger$ creates a particle with momentum $\vec{p}$

[2] Explicitly, for $\vec{p} = 0$, we can define

$$\epsilon_1^\mu = (0, 1, 0, 0) \qquad \epsilon_2^\mu = (0, 0, 1, 0) \qquad \epsilon_3^\mu = (0, 0, 0, 1) \ .$$

For general $\vec{p}$, we can take

$$\epsilon_r^\mu(\vec{p}) = \Lambda_{\vec{p}}{}^\mu{}_\nu \epsilon_r^\nu(\vec{p} = 0) \ ,$$

where $\Lambda_{\vec{p}}$ is the boost that takes a particle of mass m from momentum zero to momentum $\vec{p}$.

[3] To do this, we can demand that the classical relation $(\pi_i = \partial_0\phi_i - \partial_i\phi_0 = \partial_0\phi_i + \frac{1}{m^2}\partial_i(\nabla \cdot \pi)$ holds as an operator expression. The result is

$$\pi^i(x) = \int \frac{d^d p}{(2\pi)^d} \frac{iE_p}{2} (\delta_{ij} - \frac{p_i p_j}{p^2 + m^2})(-a_{\vec{p}}^r \epsilon_r^j e^{-ip\cdot x} + (a_{\vec{p}}^r)^\dagger \epsilon_r^j e^{ip\cdot x})$$

and energy $E_p = \sqrt{\vec{p}^2 + m^2}$. Thus, in contrast to the scalar field theory, we have three independent particle states for each momentum. To understand the physical significance of these, we can consider the particles with zero momentum and look at the action of the rotation operator. We find that the three states transform under rotations in precisely the same way as three states of a spin 1 particle, where we can make the identifications

$$a_3^\dagger|0\rangle \sim |j = 1 m = 0\rangle \qquad (a_1^\dagger \pm i a_2^\dagger)|0\rangle \sim |j = 1 m = \pm 1\rangle \; .$$

Thus, we conclude that the general free vector field theory with mass parameter m describes the physics of spin 1 particles of mass m. We will have more to say about the case $m = 0$ later on.