Scalar field theory II: time-dependent fields and correlation functions

To start, let's remind ourselves what physical systems we are describing with scalar fields:

Q: What real-world particles are described by scalar fields?

Answer: While we haven't discovered any fundamental particles with zero spin, the Standard Model predicts the existence of a Higgs particle with zero spin. However, there are many composite particles that are spinless. These include pions (bound states of two quarks) and other exotic unstable particles, but more familiar particles such as α *particles or even Helium atoms. As long as we are considering a physical process where the energy is not sufficient to excite the internal degrees of freedom of a composite particle, then the description of the composite particle in terms of a single field is appropriate.*

We finished last time by pointing out that the state $\phi(x_0)|0\rangle$ can be interpreted as describing a particle localized at some point $x = x_0$ at a specific time, which we can call $t = 0$.[1] Thus, we can say the operator $\phi(x_0)$ creates a particle at $x = x_0$ at time $t = 0$.[2] From elementary quantum mechanics, we know that localized particle states spread out quickly with time, so at any other time, the particle will not be localized. It is therefore helpful to ask:

Q: How do we create a particle at $x = x_0, t = t_0$?

Answer: We can perform this operation in three steps. We first translate the state backwards in time from $t = t_0$ *to* $t = 0$*, then create the particle at* $x = x_0$*, and finally translate forward in time from* $t = 0$ *to* $t = t_0$*. This combination of operations is achieved by the operator*

$$\phi(x,t) \equiv e^{iHt}\phi(x)e^{-iHt}$$

where we are using the fact that the Hamiltonian H *is the operator associated with time translations.*[3]

The quantity $\phi(x,t)$ gives us a quantum operator for every choice of x and t. It is an example of a HEISENBERG PICTURE operator in quantum mechanics. Generally, the expectation value of any operator $\mathcal{O}$ in a time-dependent state can be rewritten as the expectation value of the corresponding Heisenberg picture operator $\mathcal{O}(t)$ in the

[1] As in ordinary quantum mechanics, such localized states are not physically realizable since they are superpositions involving momentum eigenstates with arbitrarily large energy.

[2] Since $\phi(x)$ also includes terms with annihilation operators, we can say more precisely that $\phi(x)$ is the linear combination of an operator that creates a particle at $x = x_0$ and an operator that annihilates a particle at $x = x_0$.

[3] Note that we usually think of e^{-iHt} as the operator that evolves a state forward in time. However, this is equivalent to saying that we translate the state back in time, since time evolution is the operation by which events that were in the future are now in the present.

state at $t = 0$:

$$\begin{aligned} \langle \psi(t)|\mathcal{O}|\psi(t)\rangle &= \langle \psi(0)|e^{iHt}\mathcal{O}e^{-iHt}|\psi(0)\rangle \\ &\equiv \langle \psi(0)|\mathcal{O}(t)|\psi(0)\rangle \end{aligned}$$

The first way of writing things is generally known as the SCHRÖDINGER picture, while the reformulation using time-dependent operators and states that don't depend on time is called the Heisenberg picture. From the definition, it follows that any Heisenberg picture operator satisfies

$$\frac{d\mathcal{O}(t)}{dt} = i[H, \mathcal{O}(t)] \; . \tag{1}$$

As an example, let's work out the following:

Q: What is $a^\dagger_{\vec{p}}(t)$?

Answer: Using the result (1), we have

$$\begin{aligned} \frac{da^\dagger_{\vec{p}}(t)}{dt} &= i[H, a^\dagger_{\vec{p}}(t)] \\ &= i[H, e^{iHt}a^\dagger_{\vec{p}}e^{-iHt}] \\ &= iHe^{iHt}a^\dagger_{\vec{p}}e^{-iHt} - ie^{iHt}a^\dagger_{\vec{p}}e^{-iHt}H \\ &= ie^{iHt}Ha^\dagger_{\vec{p}}e^{-iHt} - ie^{iHt}a^\dagger_{\vec{p}}He^{-iHt} \\ &= ie^{iHt}[H, a^\dagger_{\vec{p}}]e^{-iHt} \\ &= ie^{iHt}E_p a^\dagger_{\vec{p}}e^{-iHt} \\ &= iE_p a^\dagger_{\vec{p}}(t) \end{aligned}$$

where $E_p = \sqrt{m^2 + p^2}$. It follows that

$$a^\dagger_{\vec{p}}(t) = e^{iE_p t}a^\dagger_{\vec{p}} \; .$$

Using this basic result, we can immediately write down an expression for the time-dependent field operator,

$$\phi(x,t) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} \{a_{\vec{p}}e^{-ip\cdot x} + a^\dagger_{\vec{p}}e^{ip\cdot x}\}$$

This is almost the same as our earlier expression, but now in the exponentials, we have $p \cdot x = E_p t - \vec{p} \cdot \vec{x}$. The integral is still over only the spatial momentum, since we are simply setting $p^0 = E_p = \sqrt{m^2 + p^2}$.

This operator expression has a few nice features. First, if we reinterpret $a_{\vec{p}}$ as an ordinary complex number for each p and $a^\dagger_{\vec{p}}$ as its complex conjugate, the expression above gives the most general solution to the Klein-Gordon equation (the classical equation of motion for our scalar field. In fact, the operator expression itself satisfies the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x,t) \; .$$

Also, it is simple to check that the time-dependent version of the momentum operator $\pi(x,t) \equiv e^{iHt} E_p \pi(x) e^{-iHt}$ is related to the time-dependent field operator simply by

$$\pi(x,t) = \dot{\phi}(x,t)$$

Correlation functions

What physical properties of a quantum field theory are we interested in? So far, we've discussed the spectrum of energies and the physical properties of single particle states. Another interesting set of observable is provided by the correlation functions. Generally, these are expectation values of the form

$$\langle \phi | \mathcal{O}_1(x_1^\mu) \cdots \mathcal{O}_n(x_n^\mu) | \phi \rangle \ ,$$

where $\mathcal{O}_i$ are operators built out of the fields. We've already seen some examples of this form; for the simple one-dimensional field theory, you calculated

$$\langle 0 | \phi^2(x,0) | 0 \rangle$$

the average displacement-squared of the field at position x in the ground state. More generally, an expectation value

$$\langle 0 | \phi(x_1, 0) \phi(x_2, 0) | 0 \rangle \tag{2}$$

tells us about the correlation between the value of the field at two spatially separated points. A given quantum state involves a superposition of states with various possible field configurations (the field eigenstates). If we look at the ground state of the field theory, configurations in which the field varies quickly in space are suppressed since these are associated with higher energy. Thus, if x_1 and x_2 are very close together, we expect that $\phi(x_1, 0)$ and $\phi(x_2, 0)$ will have similar values for configurations that make a significant contribution to the ground state. If x_1 and x_2 are very far apart, there is no reason for their values to be closely correlated, so we expect that (2) should be a decreasing function of $|x_1 - x_2|$.

We can also look at correlations in time, or more generally, between the field at one point in space and time and another point in space and time:

$$\langle 0 | \phi(x_1^\mu) \phi(x_2^\mu) | 0 \rangle \tag{3}$$

For a free field theory, it turns out that any more complicated correlation functions can all be expressed in terms of this simple two-point correlation function. Thus, it will be useful to have an explicit expression for this.

Q: Evaluate this correlation function as a function of x_1^μ and x_2^μ.

Answer: We use the fact that any annihilation operator acting on the vacuum state gives zero $a_{\vec{p}}|0\rangle = 0$, *which implies also that* $\langle 0 | a_{\vec{p}}^\dagger = 0$. *Thus, we can ignore the terms* *in $\phi(x_1)$ involving creation operators and the terms in $\phi(x_2)$ involving annihilation operators. We get*

$$
\begin{aligned}
\langle 0|\phi(x_1^\mu)\phi(x_2^\mu)|0\rangle &= \langle 0|[\int \frac{d^dp}{(2\pi)^d}\frac{1}{\sqrt{2E_p}}a_{\vec{p}}e^{-ip\cdot x_1}][\int \frac{d^dq}{(2\pi)^d}\frac{1}{\sqrt{2E_q}}a^\dagger_{\vec{q}}e^{iq\cdot x_2}]|0\rangle \\
&= \int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{\sqrt{4E_pE_q}}e^{-ip\cdot x_1}e^{iq\cdot x_2}\langle 0|a_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle \\
&= \int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{\sqrt{4E_pE_q}}e^{-ip\cdot x_1}e^{iq\cdot x_2}(2\pi)^d\delta(p-q) \\
&= \int \frac{d^dp}{(2\pi)^d}\frac{1}{2E_p}e^{-ip\cdot(x_1-x_2)} \\
&\equiv D(x_1-x_2)
\end{aligned}
$$

The correlation function $D(x)$ must be Lorentz-invariant, $D(\Lambda x) = D(x)$, since by replacing x_i by Λx_i, we are describing the scalar field correlation function in the same state, at the same physical points in space and time, but as observed by someone in another reference frame. Since by definition, a scalar field at a particular spacetime point has the same value for in any reference frame, the correlation function cannot be changed by the change in reference frame.[4]

The Lorentz-invariance of $D(x)$ has an important implication for the commutator of fields at different spacetime points. Since $\phi(x^\mu)$ is linear in creation and annihilation operators, the commutator $[\phi(x_1^\mu), \phi(x_2^\mu)]$ is an ordinary function (i.e. not an operator) since the commutators of creation and annihilation operators give ordinary functions. Thus, we have

$$
\begin{aligned}
[\phi(x_1^\mu), \phi(x_2^\mu)] &= \langle 0|[\phi(x_1^\mu), \phi(x_2^\mu)]|0\rangle \\
&= D(x_1-x_2) - D(x_2-x_1)
\end{aligned}
$$

If $x_2 - x_1$ is spacelike, there will be some Lorentz transformation Λ such that $x_1 - x_2 = \Lambda(x_2 - x_1)$.[5] In this case, it follows that

$$[\phi(x_1^\mu), \phi(x_2^\mu)] = D(\Lambda(x_2-x_1)) - D(x_2-x_1) = 0$$

The vanishing of field commutators for spacelike separations is a necessary property for the physics of the field theory to be CAUSAL (i.e. for events to cause other events only in their future light cone). If the commutator did not vanish for spacelike separation, then measurements of the field at event x_1 could interfere with measurements of the field at event x_2 (e.g. repeated measurement of the field at x_1 would not have to give the same result if the field were also being measured at x_2). This violate the notion

[4]Mathematically, we can show that $D(x) = \int_{p^0>0}\frac{d^{d+1}p}{(2\pi)^d}\delta(p^2+m^2)e^{ip\cdot x}$. In this form, it is clear that the expression is Lorentz-invariant, since a transformation on $x \to \Lambda x$ can be reversed by a change of variables $p \to \Lambda p$.

[5]For example, we can boost to a frame where x_1 and x_2 occur at the same time, then do a rotation to reverse the sign of the displacement vector, then boost back to the original frame.

of CAUSALITY in physics, that events can only influence other events inside their forward light cone (i.e. such that a light signal could propagate from one point to the other).