Scalar field theory

We now have the tools to write down the most general Poincar/'e invariant local field theories involving vector, tensor, and scalar fields. While we've already looked at examples of scalar field theories, it will be useful now to go back and understand these more fully. Let's start with a question:

Q: Write the most general action scalar field for a Poincar/'e invariant local field theory of single scalar field $\phi(x)$ with linear equations of motion that determine the future evolution in terms of $\phi(\vec{x}, t = 0)$ and $\dot{\phi}(\vec{x}, t = 0)$. We require also that the classical energy is bounded from below.

Answer: To ensure linear equations of motion, we need an action that is at most quadratic in the field ϕ. To ensure that the classical evolution is determined in terms of the initial configuration and the first time derivative, the equations of motion must not include and more than two time derivatives on the field. The only possible local, Poincaré-invariant terms satisfying these conditions are:

$$\int d^d x \{ \frac{a}{2} \partial_\mu \phi \partial^\mu \phi - \frac{b}{2} \phi^2 - c\phi \} \ .$$

Here, a and b must be positive to have energy bounded from below. We can set c to zero by a redefinition of the field,[1] *$\phi \to \phi - c/b$ and set a to one by a redefinition $\phi \to \phi/\sqrt{a}$. The remaining parameter b is physical, and in previous examples, we have seen that it is related to the mass of the particles in the quantum system by $b = m^2 c^2/\hbar^2$.*

To summarize, the most general scalar field theory satisfying our physical conditions is described by an action[2]

$$S = \int d^d x \{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \} \ .$$

The classical equations of motion for this action are

$$\partial_\mu \partial^\mu \phi + m^2 \phi \equiv \ddot{\phi}^2 - \nabla^2 \phi + m^2 \phi = 0 \ ,$$

known as the Klein-Gordon equation.

The quantum theory

What is the quantum physics for this field theory? In 1+1 dimensions, we have already seen how to quantize this theory for a finite interval of space $0 \le x \le L$ and either fixed or periodic boundary conditions. To quantize in infinite space, the most rigorous approach is simply to define the theory at finite volume and then take the limit as $L \to \infty$. We will see that all physical quantities of interest remain well-defined in the

[1]If $b = 0$, the term with c is not allowed since the energy would be unbounded below.

[2]From now on, we will be using units in which masses, inverse lengths, and inverse frequencies all have the same units, chosen so that $c = \hbar = 1$.

limit. Intuitively, this should be expected by locality, since the local physics should not depend on what happens very far away (e.g. local physics should not care whether we are working with a finite system with the field constrained to vanish at the edge of the solar system or whether the field really exists all the way to infinity.)

For the theory with periodic boundary conditions on the interval $0 \le x \le L$, we saw that a basis of quantum states can be generated by acting with creation operators $a_p^\dagger$ on a vacuum state $|0\rangle$, where $a_p^\dagger$ creates a particle with momentum p and energy $\sqrt{p^2+m^2}$, and where the momenta are constrained to be $p = 2\pi n/L$. Taking $L \to \infty$, the only thing that happens is that the allowed momenta are now arbitrary. In higher dimensions, we can again start with finite ranges L_x, L_y, L_z for x, y, and z and impose periodic boundary conditions in each direction. In this case, we find that the allowed momenta for particle states are $\vec{p} = (2\pi n_x/L_x, 2\pi n_y/L_y, 2\pi n_z/L_z)$ and the corresponding energies are $E = \sqrt{\vec{p}^2 + m^2}$. Again, in the limit where the spatial region becomes infinite, the only difference is that all momenta are allowed.

For calculations, it is useful to have an expression for the field operator in terms of the operators creating particles with various momenta. In the case of a finite interval in 1+1 dimensions, we found that

$$\begin{aligned}
\phi(x) &= \frac{1}{2}\sum_{p=\frac{2\pi n}{L}} \frac{1}{\sqrt{L\omega_p}}\{a_p e^{ipx} + a_p^\dagger e^{-ipx}\} \\
\pi(x) &= \frac{1}{2}\sum_{p=\frac{2\pi n}{L}} \frac{i\sqrt{L\omega_p}}{2} i\{-a_p e^{ipx} + a_p^\dagger e^{-ipx}\}
\end{aligned}$$

The generalization to infinite volume and higher dimensions is easy enough to guess. The e^{ipx} factor multiplying a_p is simply a plane wave in the x direction. For general $\vec{p}$ in higher dimensions, the analogous factor would be $e^{i\vec{p}\cdot\vec{x}}$. Also, in the infinite volume limit, all real values of p are allowed, so the sum should become an integral. In detail, we have

$$\lim_{L\to\infty}\{\frac{2\pi}{L}\sum_{p=\frac{2\pi n}{L}} F(p)\} \to \int dp F(p)$$

Thus, up to some constant factors that we have inserted by convention, we end up with the expressions[3]

$$\begin{aligned}
\phi(x) &= \int \frac{d^d p}{(2\pi)^d}\frac{1}{\sqrt{2E_p}}\{a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\} \\
\pi(x) &= \int \frac{d^d p}{(2\pi)^d} i\sqrt{\frac{E_p}{2}}\{-a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\}
\end{aligned}$$

[3]In making the transition to the integral, there is an extra factor of L that we have absorbed into the definition of a and $a^\dagger$. Wheras the old $a_p^\dagger$ created a state $|p\rangle$ with unit norm, the new one creates a state whose norm is infinite, as usual for eigenstates labeled by a continuous variable (these have Dirac delta function normalization rather than Kronecker delta function normalization.)

If we're not completely convinced that these are the right expressions, we can simply take them as a definition of the operators a and $a^\dagger$ (it's possible to invert these equations via an inverse Fourier tranform) and check that these operators satisfy the right properties. In particular, the relation

$$[\phi(\vec{x}_1), \pi(\vec{x}_2)] = i\delta^d(\vec{x}_1 - \vec{x}_2)$$

which we proved for the finite interval must hold for infinite space as well, since it is unchanged in the limit $L \to \infty$. Using this, and the formula relating $\phi(x)$ and $\pi(x)$ to $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$, we find that

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^d \delta^d(\vec{p} - \vec{q}) \ .$$

Further, we can check that the expressions for the energy and momentum operators (which we derive based on the translation symmetries) are

$$\mathcal{E} = \int d^dx \{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\} \quad \to \quad H = \int \frac{d^dp}{(2\pi)^d} E_p a^\dagger_{\vec{p}} a_{\vec{p}} + \text{infinite constant}$$

$$\vec{\mathcal{P}} = -\int d^dx \{\dot{\phi}\nabla\phi\} \quad \to \quad \vec{P} = \int \frac{d^dp}{(2\pi)^d} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}}$$

Q: Verify that the state $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$ is a momentum eigenstate.

Thus, as we expected, $a^\dagger_{\vec{p}}$ creates a particle of momentum p. We can also construct the angular momentum operator and verify that the state with $\vec{p} = 0$ is invariant under rotations and therefore corresponds to a spin zero particle. This also follows directly from the fact that there is only a single particle state for each value of momentum.

The interpretation of $\phi(x)$

Though we have a good physical interpretation for the creation and annihilation operators, it may be less clear how to interpret the field operator $\phi(x)$ itself. To understand this, consider first the question:

Q: If we restrict to momenta much less than the particle mass, we expect that ordinary quantum mechanics should be a good description of the particles. Thus, we should be able to have particle states with any properly normalized position-space wavefunction $\psi(x)$. How can we write such a state using the quantum field theory language?

Answer: We know that $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$ is an eigenstate of momentum. Also, we know that a particle state with a given position space wavefunction has a momentum space wavefunction $\tilde{\psi}(p) = \int d^dx e^{-i\vec{p}\cdot\vec{x}}\psi(x)$. The momentum wavefunction tells us what linear combination of momentum eigenstates corresponds to the state. Thus, the state with position state wavefunction $\psi(x)$ should be

$$\int d^dp \tilde{\psi}(p)|\vec{p}\rangle = \int d^dp \int d^dx e^{-i\vec{p}\cdot\vec{x}}\psi(x) a^\dagger_{\vec{p}}|0\rangle \ .$$

On the other hand, for $|\vec{p}| \ll m$ we have $E_p \approx m$ so

$$\begin{aligned} \int \psi(x)\phi(x)|0\rangle &= \int d^d x \psi(x) \int d^d p \frac{1}{\sqrt{2E_p}} e^{-i\vec{p}\cdot\vec{x}} a^\dagger_{\vec{p}} |0\rangle \\ &= \\ intd^d p \int d^d x e^{-i\vec{p}\cdot\vec{x}} \psi(x) a^\dagger_{\vec{p}} |0\rangle \end{aligned}$$

in agreement with the state we wrote down above so long as the the momenta contributing to the quantum superposition are all non-relativistic. In non-relativistic quantum mechanics, we say that the state with wavefunction $\psi(x)$ is $\int d^d x \psi(x)|x\rangle$, so we can make the identification $\phi(x)|0\rangle \sim |x\rangle$ and interpret the field operator $\psi(x)$ as creating a particle at position x. Since $\phi(x)$ also contains an annihilation part, it also annihilates a particle at position x.

In the discussion above, we have in mind the non-relativistic limit of the field theory. However, it is true more generally that the field operator $\phi(x)$ produces a state localized at x. For example, in the theory of a complex scalar field, we have a conserved charge, and we can ask about the charge density in the state $\phi(x)|0\rangle$. In this case, we find that all charge is localized at the point x.