Constructing field theories

We now have all the tools we need to build candidate field theories for physical systems of interest and, at least in the case of non-interacting field theories, to analyze the physical properties of the particle states that arise quantum mechanically.

In general, the procedure goes like this:

- Identify the degrees of freedom and symmetries (or conserved quantities) of the physical system you are trying to describe.
- Write a general action with the appropriate number of fields and the desired symmetries. This may include some undetermined coefficients.
- Analyze the quantum states of the theory, choosing the undetermined coefficients so that the field theory predictions match the desired physics.

Of course, for many physical systems, we already know the right field theory, and we're mainly interested in doing calculations to make various predictions. However, we would like to understand where these field theories come from and why we should be convinced they are correct.

We'll start with very simple systems, consisting of free (non-interacting particles). The absence of interactions will be ensured by assuming linear equations of motion at the classical level, or a quadratic action. We've already seen that a simple quantum field theory of this type describes particles. But we'd now like to know if any system of non-interacting particles can be described by a field theory.

Classifying Particles

The first natural question to ask is: what kinds of particles are there? This leads immediately to:

Q: What physical properties distinguish different kinds of particles from each other?(By a particle, we don't necessarily mean an elementary particle, only a particle whose possible internal structure we can ignore (e.g. because the energy scales we are interested in are not large enough to probe any internal structure).

Answer: Mass, spin (or helicity), charge (electric and other types), and statistics (does the particle obey the Pauli exclusion principle or not).

Except for the last property, all of these have to do with the values of conserved quantities for the particle (mass = total energy in rest frame, spin = total angular momentum, charge = some conserved quantity related to an "internal" symmetry.)

There is a very precise mathematical way to describe this classification of particles based on conserved quantities. We first recall that in quantum mechanics, the operators corresponding to conserved quantities are the same operators that generate (i.e. give the infinitesimal transformations for) the associated symmetries. Thus, asking about the values of conserved quantities for a particle is closely related to asking about how symmetry transformations act on the quantum states of that particle. Mathematically, we can say that the quantum states describing a single particle are in some REPRESENTATION of the group of symmetries, and this representation is what distinguishes different types of particles from one another.

Lets look at a simple example. Consider a quantum state describing a particle in its rest frame. Now consider what happens when we act on this state with rotation operators. Since the conserved quantity associated with rotations is angular momentum, the operators that give the change in the state if we make an infinitesimal rotation around the x, y, or z axes are the angular momentum operators J_x, J_y, J_z. We know from basic quantum mechanics that the action of these operators is different depending on the spin of the particle: for spin 0, we have $J_i|\psi\rangle = 0$, while for higher spins, the angular momentum operators take us between different spin states of a particle (e.g. J_+ and J_- take us between states with different J_z eigenvalues). Thus, the spin of a particle has to do with how its states behave when we act with rotation generators. The quantum states of a single particle at rest form a *representation* of the rotation group, and spin is a label for the different possible representations.

More generally, the different types of particles are classified by how they behave under the action of the whole group of symmetries for the theory, which can include space and time translations, rotations, boosts (for a relativistic theory), and symmetries whose conserved quantities are the various charges (e.g. electric charge, colour charge) of the theory.

Classifying Fields

We would now like to understand what kinds of fields we need to describe the different kinds of particles. Since a field is classically just some function of space and time, it would appear at first that one field is the same as any other field. However, just as the different kinds of particles could be classified by their behavior under the action of symmetry transformations, we can have different kinds of fields, depending on how they behave when we act with (classical) symmetry transformations. Let's look at a couple of examples:

Q: Suppose we have a single field $\phi(x, y, t)$ in a theory with rotational symmetry in the $x - y$ plane. If we apply a 90 degree clockwise rotation, what is an expression for the new field in terms of the old field?

Answer: $\tilde{\phi}(x, y, t) = \phi(-y, x, t)$.

Q: Now suppose we have two fields $\phi_1(x, y, t)$ and $\phi_2(x, y, t)$ in a theory with rotational symmetry in the $x - y$ plane. What happens to these if we do a 90 degree clockwise rotation?

Answer: The simplest transformation rule would be

$$\begin{aligned} \tilde{\phi}_1(x, y, t) &= \phi_1(-y, x, t) \\ \tilde{\phi}_2(x, y, t) &= \phi_1(-y, x, t) \ . \end{aligned}$$

Q: Finally, suppose our two fields are the x and y components of the electric field $E_x(x, y, t)$ and $E_y(x, y, t)$. What happens to these if we do a 90 degree clockwise rotation?

Answer: Here, we have

$$\begin{aligned} \tilde{E}_x(x, y, t) &= E_y(-y, x, t) \\ \tilde{E}_y(x, y, t) &= -E_x(-y, x, t) \; . \end{aligned}$$

In this case, the rotation acts on the coordinates as before, but it also mixes up the field components (i.e. we need to rotate both the space and the electric field vector). This is what it means for a field to be a VECTOR FIELD. In the previous example, we had the same number of field components, but the components didn't mix. Here, we just have two separated SCALAR FIELDS.

For more general rotations $x_i \to R_{ij} x_j$, we can write the transformation rule for a vector field as

$$\phi_i(\vec{x}, t) = R_{ij} \phi_j(R^{-1} \vec{x}, t)$$

while the transformation rule for a scalar field is simply

$$\phi(\vec{x}, t) = \phi(R^{-1} \vec{x}, t) \; .$$

The R^{-1} inside is natural since in order to determine the new field at a given point, we want to do the opposite of our rotation to find which point to evaluate the old field at.

There are more general possibilities, known as TENSOR FIELDS, that transform as

$$\phi_{i_1 \cdots i_n}(\vec{x}, t) = R_{i_1 j_1} \cdots R_{i_n j_n} \phi_{j_1 \cdots j_n}(R^{-1} \vec{x}, t)$$

As we will see, there is a systematic way to figure out what all the possible transformation rules are for a given set of symmetries. Mathematically, this is the subject of REPRESENTATION THEORY.

Building invariant actions.

We are now in a better position to understand step 2 of our general procedure for coming up with field theories to describe physical systems, "Write a general action with the appropriate number of fields and the desired symmetries."

To determine the possible field theory actions with a given set of symmetries, the first step will be to figure out the possible transformation rules for the fields. Once we know how the fields transform, it will be straightforward to figure out what kinds of actions that we can write down that will be invariant under the symmetries.

As a quick example, if we have two scalar fields ϕ_1 and ϕ_2, a term $\int dt dx dy (m^2 \phi_1^2 + M^2 \phi_2^2)$ would be invariant under rotational symmetry. But if ϕ_1 and ϕ_2 are components of a vector field, such a term would only be invariant under rotations if $m = M$.

Since both particle states and fields can be classified by how they behave under the action of symmetry transformations, it is plausible that by considering all the different types of fields, we will be able to describe all the different types of particles.