

Symmetries and conservation laws in quantum mechanics

Using the action formulation of local field theory, we have seen that given any continuous symmetry, we can derive a local conservation law. This gives us classical expressions for the density of the conserved quantity, the current density for this, and (by integrating the density over all space) the total conserved quantity (often referred to as the CHARGE). All of these observables can be promoted to quantum operators by writing them in terms of the field variables and their corresponding momenta (e.g. the ϕ_n s and the corresponding p_n s for our guitar string field theory). But as we will see today, in quantum mechanics there is an even more direct connection between the conserved quantities and the symmetries: the operator for the total conserved quantity (i.e. the charge) is *the same* operator that gives the change in the state under an infinitesimal version of the associated symmetry. For example, if \mathbf{P} is the momentum operator (associate with translation invariance), the small change in a state $|\psi\rangle$ under an infinitesimal translation by an amount δx is

$$\delta|\psi\rangle = \frac{1}{i\hbar}\mathbf{P}|\psi\rangle$$

Conserved quantities in quantum mechanics

Let's begin by asking a question:

What is meant by a conserved quantity in quantum mechanics?

Answer:

Since observables do not have definite values for general states, it is not clear what we mean by saying that an observable is independent of time for general states. We know that there are some states, the eigenstates $|\lambda_i\rangle$ for the observable, that have definite values. If \mathcal{O} is a conserved quantity and $|\lambda_i\rangle$ has the definite value λ_i for \mathcal{O} , we should certainly demand that the state continues to have this definite value under time evolution. This requires that H acting on $|\lambda_i\rangle$ (proportional to the infinitesimal change in the state under time evolution) is again $|\lambda_i\rangle$ ¹ In other words, $|\lambda_i\rangle$ must also be an eigenstate of the Hamiltonian. Since the Hamiltonian and the observable \mathcal{O} have a common set of eigenstates, they are both represented by diagonal matrices in this basis, so we conclude that the Hamiltonian H and the Hermitian operator \mathcal{O} for a conserved observable must commute with each other,

$$[H, \mathcal{O}] = 0 .$$

So far, we have only demanded that states with a definite value for \mathcal{O} retain this value. However, as a consequence of this condition, we can show that for any state, the

¹In cases where there are multiple states with the same eigenvalue, we only require that H maps each state with a given eigenvalue to another with the same eigenvalue. However, by a change of basis, we can ensure that H maps the eigenstates to themselves.

probabilities P_i for finding the eigenvalue λ_i in a measurement of \mathcal{O} are independent of time.² To see this, we note that an arbitrary state can be expanded as

$$|\psi(t=0)\rangle = \sum_i c_i |\lambda_i\rangle$$

where the probability for measuring λ_i is $|c_i|^2$. As we have seen, the $|\lambda_i\rangle$ must be energy eigenstates, so we must have

$$|\psi(t)\rangle = \sum_i c_i e^{-iE_i t/\hbar} |\lambda_i\rangle .$$

The probability for measuring λ_i is now $|c_i e^{-iE_i t/\hbar}|^2$, just as before.

Summary

A conserved quantity in quantum mechanics can be defined as one for which the probabilities of measuring the various eigenvalues for that quantity are independent of time in all states. Mathematically conserved quantities correspond to Hermitian operators \mathcal{O} satisfying

$$[H, \mathcal{O}] = 0 .$$

Symmetries in quantum mechanics

Now that we understand conserved quantities...

What is meant by a symmetry in quantum mechanics?

Answer:

A symmetry is a transformation on the space of physical configurations, so in quantum mechanics, it must be represented by a map from the Hilbert space to itself. The transformation should map states to physically equivalent states, so the inner product between a pair of transformed states must be the same as the inner product between the original states (recall that the inner product contains the basic information about probabilities for measurements). This requires that the symmetry transformation is a linear, UNITARY operator acting on the Hilbert space,³ that is, a map

$$|\psi\rangle \rightarrow \mathcal{U}|\psi\rangle$$

such that

$$\mathcal{U}^\dagger \mathcal{U} = 1 .$$

Not all unitary operators are symmetries, however. Recall that in classical mechanics, we required that symmetries map solutions to the equations of motion to other solutions

²We might take this as a basic definition of what we mean by a conserved quantity.

³If we were talking about a real vector space, the equivalent statement would be that if we want to preserve the dot product, we need an orthogonal transformation.

of the equations of motion. In quantum mechanics, the analogous statement would be that a symmetry maps a solution of the Schrodinger equation to another solution of the Schrodinger equation. To find a mathematical expression for this condition, we require that $\mathcal{U}|\psi\rangle$ satisfies the Schrödinger equation whenever $|\psi\rangle$ does. Assuming that \mathcal{U} doesn't depend explicitly on time, the left side of the Schrödinger equation for \mathcal{U} is

$$i\hbar \frac{d}{dt}(\mathcal{U}|\psi\rangle) = i\hbar \mathcal{U} \frac{d}{dt}|\psi\rangle = \mathcal{U}H|\psi\rangle$$

while the right side is

$$H\mathcal{U}|\psi\rangle .$$

So our condition is equivalent to $\mathcal{U}H = H\mathcal{U}$ or $[\mathcal{U}, H] = 0$.

Summary

A symmetry is a linear unitary transformation on the Hilbert space which give solutions to the Schrodinger equation when acting on solutions to the Schrodinger equation. Mathematically, this can be expressed as $[\mathcal{U}, H] = 0$.

Infinitesimal symmetries in quantum mechanics

Given any continuous symmetry, $\mathcal{U}(a)$, we can consider an infinitesimal version by making a very small (assuming that $a = 0$ gives the identity operator). In this case,⁴

$$\mathcal{U}(\epsilon)|\psi\rangle = |\psi\rangle + \epsilon \frac{1}{i\hbar} \mathcal{O}|\psi\rangle + (\text{order } \epsilon^2) \quad (1)$$

or equivalently

$$\delta|\psi\rangle = \epsilon \frac{1}{i\hbar} \mathcal{O}|\psi\rangle .$$

Here, we have defined an operator \mathcal{O} by

$$\mathcal{O}|\psi\rangle = i\hbar \lim_{\epsilon \rightarrow 0} (\mathcal{U}(\epsilon) - 1)|\psi\rangle / \epsilon$$

Since $\mathcal{U}^\dagger \mathcal{U} = 1$ and since $\mathcal{U}(\epsilon) = 1 + \epsilon \frac{1}{i\hbar} \mathcal{O} + (\text{order } \epsilon^2)$, it follows that $\mathcal{O} = \mathcal{O}^\dagger$, so \mathcal{O} is Hermitian.⁵ Furthermore, the the order ϵ term in $[\mathcal{U}(\epsilon), H] = 0$ gives $[\mathcal{O}, H] = 0$.

Summary

Given any continuous symmetry, we can associate a Hermitian operator \mathcal{O} satisfying $[\mathcal{O}, H] = 0$. The operator \mathcal{O} is the GENERATOR of the symmetry; acting on a state it gives the small change in the state under the infinitesimal version of the symmetry.

⁴The factors of i and \hbar are conventions that will be explained below.

⁵The factor of i in the definition of \mathcal{O} above ensures that \mathcal{O} is Hermitian rather than antihermitian.

Conclusion: Symmetries and Conserved quantities

We have seen that conserved quantities in quantum mechanics are represented by Hermitian operators \mathcal{O} that commute with the Hamiltonian. On the other hand, any such operator is the generator of a symmetry. Thus, it is manifestly true in quantum mechanics that symmetries are associated with conserved quantities. The operator for the conserved quantity *is* the operator that generates the corresponding symmetry. Note that the factor of \hbar in the definition of \mathcal{O} in (1) was chosen so that the observable \mathcal{O} associated with a symmetry takes its usual definition (e.g. so that the operator associated with translations is momentum rather than \hbar times momentum).