Analyzing field theories from an action

The formulation of physics in terms of an action principle is both elegant and powerful. The action allows us to derive the equations of motion, to determine the symmetries, to calculate the conserved quantities associated with these symmetries, and to systematically make the transition to quantum mechanics.

Today, we'll get some practice analyzing field theories from an action principle. To begin, consider the action

$$S = \int dt \int_0^L dx \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \beta \left(\frac{\partial^2 \phi}{\partial x^2} \right)^2 \right\}$$

What are the equations of motion for the field ϕ?

Answer: $\frac{\partial^2 \phi}{\partial t^2} = -\beta \frac{\partial^4 \phi}{\partial x^4}$

As you've read, symmetries are transformations

$$\phi_i \to \tilde{\phi}_i$$

that leave the action unchanged. Roughly, these are maps between physically "equivalent" (though still distinct) configurations of the system.

As an example, consider the field theory

$$S = \int dt \int_0^L dx \left\{ \frac{1}{2} \rho \left(\frac{\partial \phi_y}{\partial t} \right)^2 + \frac{1}{2} \rho \left(\frac{\partial \phi_z}{\partial t} \right)^2 - \frac{1}{2} \tau \left(\frac{\partial \phi_y}{\partial x} \right)^2 - \frac{1}{2} \tau \left(\frac{\partial \phi_z}{\partial x} \right)^2 \right\}$$

which describes a string that can oscillate in either of two directions.

What are some symmetries of this field theory?

Answer: Examples include time translation invariance $\tilde{\phi}_i(x,t) = \phi_i(x, t+a)$ *and rotations that mix the* ϕ_y *and* ϕ_z *components of the field,* $\tilde{\phi}_i = R_{ij}\phi_j$*, where* R *is a two-dimensional rotation matrix.*

Perhaps the most important reason to identify symmetries of a physical theory is that they are associated with conserved quantities (Noether's theorem). For example, translation invariance is associated with momentum conservation, time translation invariance is associated with energy conservation, and rotational invariance is associated with angular momentum conservation. In some sense, the most fundamental definition of these physical quantities is that they are the conserved quantities associated with the various symmetries.

Generally speaking, a conserved quantity is a physical quantity that does not change under time evolution. However, we can often make a more powerful statement, known as a LOCAL CONSERVATION LAW. For example, energy conservation says that the total energy of a system with time-translation invariance does not change with time. However, in nature energy is also conserved locally: for any spatial region, the change in energy of the region in a time ΔT is equal to the amount of energy that flows into the region during that time interval.

How can we express this relation mathematically using integral expressions, in terms of the energy density ρ and the energy current $\vec{S}$? What is the differential version of this?

Answer: for any region V, we must have $\frac{d}{dt}\int_V \rho = -\int_{\partial V} \vec{S} \cdot d\vec{A}$, or in differential form, $\partial_t \rho = \nabla \cdot \vec{S}$

It turns out that this more powerful type of conservation law follows automatically when the system under consideration is a LOCAL FIELD THEORY, that is, a field theory that can be written as

$$S = \int dt \int d^d x \mathcal{L}(\phi_i(x,t))$$

where the LAGRANGIAN DENSITY $\mathcal{L}$ is an algebraic function of the field and its derivatives (a finite number of derivatives). The argument parallels the one showing that we have a simple conservation law associated with any symmetry:

Argument that symmetries of a local field theory give local conservation laws

- Suppose we have a local field theory with a symmetry whose infinitesimal form can be written (here x represents all the spatial coordinates)

$$\delta\phi_i(x,t) = \epsilon\beta_i(x,t)$$

  where β_i might depend on the fields and their derivatives.

- Now consider a more general transformation where ϵ depends on space and time.

$$\delta\phi_i(x,t) = \epsilon(x,t)\beta_i(x,t) \tag{1}$$

  This is NOT A SYMMETRY of the action (since the right side is now completely arbitrary), BUT the variation of the action should vanish when ϵ is constant, so it should be possible to write the first order variation of the action as

$$\delta S = \int dt d^d x \partial_\mu \epsilon J^\mu \tag{2}$$

  where the index μ runs over all space coordinates and time and J^μ are some expressions written in terms of the field.

- Now, suppose a field $\phi(x,t) = \phi^{EOM}(x,t)$ satisfies the equation of motion. Then *any variation* of the action about this field should vanish. Since the variation takes the form (2) for variations of the form (1), it must be that for any choice $\epsilon(x,t)$,

$$\delta S = \int dt d^d x \partial_\mu \epsilon J^\mu(\phi^{EOM}) = 0$$

where we have used the superscript ^{EOM} to indicate that ϕ^{EOM} satisfies the equations of motion.

- Choosing ϵ to vanish at the boundaries, we can integrate by parts to conclude that for any ϵ (that vanishes at the boundaries)

$$\int dt d^d x \epsilon \partial_\mu J^\mu(\phi^{EOM}) = 0$$

The only way this can be true for arbitrary $\epsilon(x,t)$ is if

$$\partial_\mu J^\mu(\phi^{EOM}) = 0$$

Recalling that μ runs over all space components and time, this equation becomes

$$\partial_t J^t = -\nabla \cdot \vec{J} ,$$

which is exactly the form of a local conservation law.

- In practice, we need only consider the variation (1), write the first order change in the action in the form (2), and read off the J^t (the density of the conserved quantity) as the expression coupling to $\partial_t \epsilon$, and $\vec{J}$, (the current density for the conserved quantity) as the expression coupling to the spatial derivatives of ϵ. The total conserved quantity (often referred to as the CHARGE) is given by integrating J^t over all of space,

$$Q = \int d^d x J^t .$$

Procedure for finding currents in practice:

1. Write the infinitesimal symmetry in the form[1]

$$\delta \phi_i(x,t) = \epsilon \beta_i(x,t)$$

2. Calculate $S(\phi_i(x,t) + \epsilon \beta_i(x,t))$ and find the term linear in ϵ (this is δS)

[1]To find this: if the symmetry transformation depends on a parameter a, figure out the value a_0 of a which corresponds to no change at all, write the transformation for $a = a_0 + \epsilon$, and read off $\delta\phi$ as the term proportional to epsilon.

3. Use integration by parts to rewrite δS as[2]

$$\delta S = \int dt \int d^d x \partial_\mu \epsilon J^\mu(\phi_i)$$

Then J^t (the quantity multiplying $\partial_t \epsilon$) is the density of the conserved quantity, while $\vec{J}$ (the quantity multiplying $\vec{\partial}_x \epsilon$) is the current of the conserved quantity.

[2] There may initially be terms of the form $\int \epsilon \mathcal{O}$ with no derivatives on ϵ, but these always have $\mathcal{O} = \partial_\mu W^\mu$ for some W, so we can integrate by parts to get $\int \epsilon \mathcal{O} = \int \epsilon \partial_\mu W^\mu = -\int \partial_\mu \epsilon W^\mu$.