Action principles

Many physical laws can be expressed in terms of an ACTION PRINCIPLE. This means that the classical evolution of the system is always such that the variables $\{q_i(t)\}$ provide an extremum for some functional $S[\{q_i(t)\}]$ known as the ACTION.

The idea is very similar to the usual notion of determining the equilibrium position for a system by minimizing the energy. For example, if we wish to find the shape of a thin rope stretched between two points, we may minimize the potential energy

$$E[y(x)] = \int dm\ g\ h = \int_0^a \left(\rho dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) gy(x)$$

subject to a constraint that the endpoints of the string and the total length of the string are fixed. We could alternatively express the minimization condition by some differential equation and boundary conditions, but the minimization idea is intuitively simpler.

In the same way, the dynamical evolution of a system may be determined by minimizing an action functional, but now the variables are functions of time, and the action involves an integral between initial and final times of some quantity known as the LAGRANGIAN

$$S[\{q_i(t)\}] = \int_{t_i}^{t_f} dt L(q_i(t), \dot{q}_i(t)) \ .$$

The action is simply a map that associates a single real number to any possibility for the time evolution of the system (described by $\{q_i(t)\}$) between t_i and t_f. The action principle tells us that the physical evolution (i.e. what actually happens) must extremize this function.

For a mechanical system the Lagrangian can be expressed as the kinetic minus the potential energy.

Example

Let's see how this works in practice. Perhaps the simplest example to consider is a single non-relativistic particle in one dimension, moving in a potential $V(x)$. The classical equations of motion for this system (starting with $F = ma$) are

$$m\ddot{x} = -V'(x) \ .$$

We will now see that this equation of motion will be satisfied if and only if $x(t)$ is an extremum of the action

$$S[x(t)] = \int_{t_i}^{t_f} dt \{ \frac{1}{2} m\dot{x}(t)^2 - V(x(t)) \} \ . \tag{1}$$

over the set of functions with fixed values x_i and x_f for some initial and final times t_i and t_f.

How do we determine whether $x(t)$ is an extremum of S? For a simple function $f(y)$, the condition for extrema is that the first derivative $f'(y)$ vanishes. But it's not completely clear how to take the first derivative of a functional with respect to a function. Going back to the simple function minimization, we can alternately say that y_0 is an extremum of $f(y)$ if

$$f(y_0 + \delta y) = f(y_0) + \mathcal{O}(\delta y^2)$$

i.e. that the linear term in the Taylor expansion of f around y_0 vanishes. Put this way, we can more easily generalize to the action case. We can say that $x_0(t)$ extremizes the action if

$$S[x_0(t) + \delta x(t)] = S[x_0(t)] + \mathcal{O}(\delta x^2) \ .$$

for all possible choices of δx. In other words, we demand that in the expansion of $S[x_0(t) + \delta x(t)]$ with respect to δx, the term linear in $\delta x(t)$ vanishes.

Starting from (9), we have

$$S[x_0(t) + \delta x(t)] = \int dt \{ \frac{1}{2} m(\dot{x}_0(t) + \delta \dot{x}(t))^2 - V(x_0(t) + \delta x(t)) \} \ .$$

The term linear in δx (which we refer to as δS) is:

$$\begin{aligned} \delta S &= \int_{t_i}^{t_f} dt \{ m\dot{x}_0(t)\delta\dot{x}(t) - V'(x_0(t))\delta x(t) \} \\ &= \int_{t_i}^{t_f} dt \partial_t \left[m\dot{x}_0(t)\delta x(t) \right] + \delta x(t) \left[-m\ddot{x}_0(t) - V'(x_0(t)) \right] \\ &= \int_{t_i}^{t_f} dt \delta x(t) \left[-m\ddot{x}_0(t) - V'(x_0(t)) \right] \end{aligned}$$

where we have integrated by parts to get to the second line and used that δx vanishes at t_i and t_f to get to the third line. The point of integrating by parts is to ensure that all terms in the final expression are proportional to δx (and not $\delta \dot{x}$).

This expression is zero for all possible $\delta x(t)$ if and only if

$$-m\ddot{x}_0(t) - V'(x_0(t)) = 0$$

which is precisely the equation of motion.

Actions for fields

For a field theory, an action should assign a real number to any classical field trajectory $\phi(x,t)$, usually subject to boundary conditions (e.g. that the field ϕ goes to zero at spatial infinity), and fixed initial and final values $\phi(x,t_i)$ and $\phi(x,t_f)$. In this case, to get a single number from a function of two (or more variables), the action is usually an integral over time and space directions,[1]

$$S = \int dt L(\phi(x,t)) = \int dt dx \mathcal{L}(\phi(x,t)) \ .$$

[1] We will see later why the action should take this form.

Here, we still refer to L as the Lagrangian, and we introduce the Lagrangian density $\mathcal{L}$, which is typically an algebraic function of ϕ and its derivatives.

As with simple mechanical systems, the action for a field theory can usually be expressed as the kinetic energy (terms in the energy involving time derivatives of the field) minus the potential energy (terms in the energy not involving time-derivatives, but possibly involving spatial derivatives). For a single field $\phi(x,t)$ in one dimension governed by the usual wave equation, this gives

$$S = \int_{t_i}^{t_f} dt \int_0^L dx \left\{ \frac{1}{2}\rho\dot{\phi}^2 - \frac{1}{2}\tau(\phi')^2 \right\}$$

where we will assume the field is constrained to vanish at $x = 0$ and $x = L$. To see that extremizing this action reproduces the wave equation, suppose that ϕ_0 extremizes the action and consider a small variation $\delta\phi$. The variation in the action about ϕ_0 is

$$S[\phi_0 + \delta\phi] = \int_{t_i}^{t_f} dt \int_0^L dx \left\{ \frac{1}{2}\rho(\dot{\phi}_0 + \delta\dot{\phi})^2 - \frac{1}{2}\tau(\phi_0' + \delta\phi')^2 \right\} .$$

The linear term in $\delta\phi$ gives

$$\begin{aligned} \delta S &= \int_{t_i}^{t_f} dt \int_0^L dx \left\{ \rho\dot{\phi}_0\delta\dot{\phi} - \tau\phi_0'\delta\phi' \right\} \\ &= \int_{t_i}^{t_f} dt \int_0^L dx \left\{ \partial_t \left[\rho\dot{\phi}_0\delta\phi\right] - \partial_x \left[\tau\phi_0'\delta\phi\right] + \delta\phi \left[-\rho\ddot{\phi}_0 + \tau\phi_0''\right] \right\} \\ &= \int_{t_i}^{t_f} dt \int_0^L dx \left\{ \delta\phi \left[-\rho\ddot{\phi}_0 + \tau\phi_0''\right] \right\} \end{aligned}$$

As above, we integrate by parts so that any derivatives appearing on $\delta\phi$ are moved to act on ϕ_0. This way, we are able to write the variation of the action as an integral over $\delta\phi$ times the quantity in square brackets. This vanishes for arbitrary $\delta\phi$ if and only if the quantity in square brackets vanishes,

$$-\rho\ddot{\phi}_0 + \tau\phi_0'' = 0$$

so we see that the action is extremized if and only if ϕ_0 must satisfies the wave equation.

Actions and symmetries

One of the most beautiful results in physics is the fundamental relation between symmetries and conservation laws.

In the language of actions, a SYMMETRY is a transformation on the basic variables (a map from trajectories to trajectories)

$$q_\alpha(t) \rightarrow \tilde{q}_\alpha(t)$$

which leaves the action unchanged:

$$S[\tilde{q}_\alpha(t)] = S[q_\alpha(t)]$$

It must leave the action unchanged regardless of whether or not $q_\alpha(t)$ satisfies the equations of motion.

For example, the action (9) is invariant under time translations

$$\tilde{x}(t) = x(t + t_0) \ . \tag{2}$$

For $V = 0$, the action would also be invariant under spatial translations

$$\tilde{x}(t) = x(t) + x_0 \ . \tag{3}$$

Exercise: What are some symmetries of the action for the wave-equation? If a particular trajectory satisfies the equations of motion, it follows that any trajectory related by a symmetry also satisfies the equation of motion.

Noether's theorem says that for any symmetry of a physical system, there is a corresponding CONSERVED QUANTITY, i.e. a quantity whose value does not change with time (assuming the equations of motion are satisfied). One of the most important uses of an action in classical physics is that it allows us to quickly derive the conserved quantity associated with any symmetry. We will focus on symmetries (such as translations or rotations) that are continuous and can be expressed in an infinitesimal form

$$\delta q_\alpha(t) = \epsilon W_\alpha(t) \tag{4}$$

where ϵ is an infinitesimal constant and W_α might be some function of the variable q_α. For example, for the translations (3), we have

$$\delta x(t) = \epsilon \tag{5}$$

so $W = 1$ while for time translations (2), we have

$$\delta x(t) = \epsilon \dot{x}(t) \tag{6}$$

so $W = \dot{x}$.

The important part

To derive the conservation law associated with a symmetry transformation expressed in the form (4), suppose that we consider a more general transformation where ϵ depends on time.

$$\delta q_\alpha(t) = \epsilon(t) W_\alpha(t) \tag{7}$$

This is not generally a symmetry of the action (since the right side is now completely arbitrary), BUT the variation of the action should vanish when ϵ is constant, i.e. when $\dot{\epsilon} = 0$. Thus, it should be possible to write the variation of the action as

$$\delta S = \int_{t_i}^{t_f} dt \dot{\epsilon}(t) \mathcal{Q}(q_\alpha(t)) \ .$$

On the other hand, if $q_\alpha(t)$ satisfies the equation of motion, then *any variation* of the action about this trajectory must vanish. So it must be that for any $\epsilon(t)$,

$$\int_{t_i}^{t_f} dt\dot{\epsilon}(t)\mathcal{Q}(q_\alpha^{EOM}(t)) = 0$$

where we have used the superscript ^{EOM} to indicate that q^{EOM} satisfies the equations of motion. Choosing $\epsilon(t)$ to vanish at the initial and final times, we can integrate by parts to obtain

$$\int_{t_i}^{t_f} dt\epsilon(t)\frac{d}{dt}\mathcal{Q}(q_\alpha^{EOM}(t)) = 0 .$$

The only way this can be true for arbitrary $\epsilon(t)$ is if

$$\frac{d}{dt}\mathcal{Q}(q_\alpha^{EOM}(t)) = 0 .$$

Thus, $\mathcal{Q}$ is a conserved quantity.

The procedure for deriving a conserved quantity

- Write the symmetry in infinitesimal form $\delta q_\alpha(t) = \epsilon W_\alpha(t)$
- Write the variation of the action under the related transformation $\delta q_\alpha(t) = \epsilon(t)W_\alpha(t)$, using integration by parts to write it in the form

$$\delta S = \int_{t_i}^{t_f} dt\dot{\epsilon}(t)\mathcal{Q}(q_\alpha(t)) \tag{8}$$

- The conserved quantity is $\mathcal{Q}$

Example

As a simple example, if we take a free particle action

$$S[x(t)] = \int_{t_i}^{t_f} dt\{\frac{1}{2}m\dot{x}(t)^2\} . \tag{9}$$

and consider the transformation (5). We now want to make ϵ time-dependent and calculate the first-order change in the action when x is changed by an amount $\delta x(t) = \epsilon(t)$. We find

$$S(x + \delta x) = \int_{t_i}^{t_f} dt\{\frac{1}{2}m(\dot{x}(t) + \dot{\epsilon}(t))^2\}$$

so

$$\delta S = \int_{t_i}^{t_f} dt\{\dot{\epsilon}(t)m\dot{x}(t)\} .$$

Comparing with (8), we see that the conserved quantity associated with the spatial translation symmetry is $\mathcal{Q} = m\dot{x}$, which we recognize as MOMENTUM.