Particle interactions

We saw last time that the mathematical description of the physics for a simple field theory was exactly the same as the description of a collection of harmonic oscillators. The field variable $\phi(x,t)$ can be decomposed into Fourier modes

$$\phi(x,t) = \sum_n \phi_n(t) sin\left(\frac{n\pi x}{L}\right)$$

and the amplitude ϕ_n of each mode is described by the equations of a harmonic oscillator with frequency $\omega_n \propto n$. Quantum mechanically, the energy in each oscillator (i.e. for each allowed wavelength) is quantized, and we can define creation and annihilation operators $a_n^\dagger$ and a_n that add or subtract one quantum of energy from the field oscillation with wavelength λ_n.

As we discovered, the quantum spectrum of this system matches exactly with our expectation for a system with arbitrary numbers of photons confined in a region $0 \leq x \leq L$. Assuming that quantizing the electromagnetic field (which is described by the same classical wave equation) will give analogous results, we now have a *derivation* of the photon picture of light. This derivation tells us that a photon is a quantum of energy in the harmonic oscillator system that describes the oscillations of the field at some particular wavelength. With this interpretation, we can say that in our field theory system:

- The creation operator $a_n^\dagger$ creates a particle with wavelength λ_n.
- The annihilation operator a_n removes a particle with wavelength λ_n.

Using the creation and annihilation operators, we can write a basis of energy eigenstates for the field theory as

$$|N_1, N_2, N_3, \cdots\rangle = (a_1^\dagger)^{N_1}(a_2^\dagger)^{N_2}\cdots|0\rangle$$

where $|0\rangle$ is the vacuum state, and the energy of this state relative to the vacuum state is

$$E - E_0 = \hbar\omega_1(N_1 + 2N_2 + 3N_3 + \ldots)\ .$$

For this simple field theory, we can say that the number of particles will not change under time evolution.

Question: why can we say this?

Because the state $|N_1, \ldots N_n\rangle$ is an energy eigenstate. Thus, its time evolution is trivial, $|\psi\rangle \to e^{-iEt/\hbar}|\psi\rangle$.

Physically, the fact that particle number (or the momenta of the particles) doesn't change stems from the fact that the wave equation we started with is linear. This means that classically, any two solutions can be superimposed to give a new solution. In particular, the classical time evolution for an initial state with two approaching wavepackets will be the superimposed time evolution of the individual wavepackets; as a result, the packets will pass right through each other.

Adding interactions

In this section, we'll see that adding non-linear terms in the wave equation (or equivalently, non-quadratic terms in the expression for the energy) will yield a theory in which the number of particles (or the momenta of the individual particles) can change in time. Consider as an example the field theory from last time, but with a new term added to the energy:

$$E_0 = \int_0^L dx\{\frac{1}{2}\rho\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\tau\left(\frac{\partial\phi}{\partial x}\right)^2\}$$

but suppose that at some time, the equations governing the system are perturbed so that new expression for the energy includes a term

$$E_1 = \int_0^L dx\{\lambda\phi^4\} .$$

This new term will result in a perturbation to the quantum Hamiltonian. Show that with the new term in the Hamiltonian, the number of particles can change under time evolution.

Time evolution is governed by the Schrödinger eqn.

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle$$

After a short time, we have:

$$|\psi(t+\delta t)\rangle = |\psi(t)\rangle - \frac{i}{\hbar}\delta t\, H|\psi(t)\rangle$$

Starting with $|\psi(t)\rangle = |N_1, N_2, \ldots\rangle$ we want to show that the right side has terms with different numbers of particles.

more space:

Here the Hamiltonian is

$$H_o + H_1 .$$

We already saw that $H_o|N_1 \ldots N_n\rangle = E|N_1 \ldots N_n\rangle$, so the $H_o|\psi\rangle$ term has the same set of particles. To figure out what $H_1|\psi\rangle$ is, we want to write this in terms of creation + annihilation operators. We have:

$$H_1 = \int_0^L dx \{\lambda \phi^4\}$$

Using $\phi(x) = \sum \sin\left(\frac{n\pi x}{L}\right)\phi_n$, we get

$$H_1 = \sum A_{klmn}\phi_k\phi_l\phi_m\phi_n \quad \text{where } A_{klmn} = \int_0^L dx \sin\left(\frac{k\pi x}{L}\right)\ldots\sin\left(\frac{n\pi x}{L}\right)$$

Each ϕ_n is the "position" operator for a harmonic oscillator, so we can write

$$\phi_n = \sqrt{\frac{\hbar}{2m\omega_n}}\left(a_n + a_n^\dagger\right)$$

Thus:

$$H_1 = \sum \tilde{A}_{klmn}(a_k + a_k^\dagger)(a_l + a_l^\dagger)(a_m + a_m^\dagger)(a_n + a_n^\dagger)$$

where $\tilde{A}_{klmn} = \sqrt{\frac{\hbar^4}{16m^4\omega_k\omega_l\omega_m\omega_n}} \cdot A_{klmn}$

Now, $H_1|\psi\rangle = H_1|N_1 \ldots N_n\rangle$ clearly has terms with more particles. For example $A_{1111} = \int_0^L dx \sin^4\left(\frac{\pi x}{L}\right) \neq 0$, so

$H_1|\psi\rangle$ has a term $|N_1+4, N_2, \ldots\rangle$.

We have seen so far that by quantizing fields, we can describe quantum systems with arbitrary numbers of particles. The particles that we obtained from quantizing the simple wave equation had the same energies as a system of massless particles (photons), and did not interact with each other. However, as we have started to see, by modifying the classical wave equation, we can describe particles with other properties:

Non-linear terms	$\Leftrightarrow$	Interacting particles
Linear term with no derivatives	$\Leftrightarrow$	Massive particles (homework)
Fields with more components	$\Leftrightarrow$	Particles with multiple states (e.g. polarizations)

Since our current understanding of nature is that all matter and light are made up of elementary particles interacting with one another, it may now be remotely plausible that quantized fields could give the right framework for a quantitative description of the physics. But how do we know what classical field theory to start with to describe particular kinds of particles with particular kinds of interactions? To describe photons, the natural guess is to quantize the electromagnetic fields, since these provide the classical description of light. But the classical description of electrons and other matter particles doesn't involve fields at all.

In order to proceed, it will be useful to have a more systematic way of writing down field theories, and some more sophisticated tools for analyzing their physical properties. The right tool for our purposes is description of field theories in terms of an ACTION PRINCIPLE.

Homework problem: to hand in Monday

a) For a string of length L fixed at both ends and described by an action

$$S = \int dt \int_0^L dx \left\{ \frac{1}{2}\rho\dot{\phi}^2 - \frac{1}{2}\tau(\phi')^2 \right\}$$

with boundary conditions $\phi(x=0) = \phi(x=L) = 0$, calculate the expected value of $\phi^2(L/2)$, the squared displacement of the center of the string, if the string is in its ground state. (Hint: try to convert your calculation into one involving creation and annihilation operators.)

b) Your result from part a) may be unsettling. It arises because of the unphysical assumption that the string can support excitations with an arbitrarily short wavelength. Redo the calculation of part a) and give the numerical result for $\sqrt{\phi^2(L/2)}$ assuming that the minimum possible wavelength is at the atomic scale ($\lambda_{min} = 10^{-10}m$). Take the tension to be 1N, the density to be $10g/m$, and the length L to be $1m$.