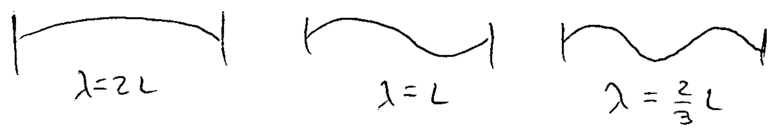
The simplest quantum field theory

In today's class, we'll analyze the simplest quantum field theory and see that its quantum states match up exactly with what we would expect from a physical system whose states include arbitrary numbers of particles.

Particles in one dimension

Imagine that we have a cavity with two ends at $x = 0$ and $x = L$. Let's try to figure out the possible energies for photons in the cavity with no momentum in the y or z directions. (In order that zero y and z momentum is allowed, we can either imagine that the cavity is very large in the y and z directions, or that we have some periodic boundary conditions in these directions.)

Assuming that the two ends of the cavity are perfect conductors so that the electric field must vanish there, what are the allowed wavelengths for classical electromagnetic plane waves in the cavity along the x direction?



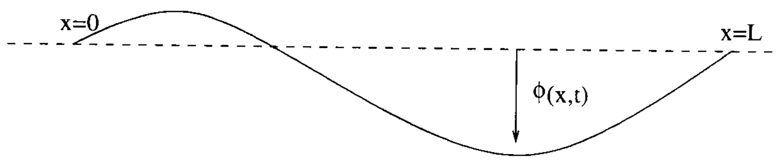
general $\boxed{\lambda_n = \frac{2L}{n}}$

Can you write a general formula for the allowed energy of a state with an arbitrary number of photons? (Hint: According to Einstein, a light wave at frequency f is composed of photons with individual energies hf.)

For wavelength λ_n, the freq. is $f_n = \frac{c}{\lambda_n}$ $\therefore$ allowed energy in light with wavelength λ_n is $E_n = N_n \cdot h f_n = \frac{hc \cdot n}{2L} N_n$ ← # of photons w. wavelength λ_n

Total allowed energy:

$$\boxed{E = \sum_n N_n \cdot \frac{hc}{2L} \cdot n}$$



The field theory

A FIELD THEORY is a physical system whose classical configurations are described by a FIELD, i.e. a function of space and time. The most familiar examples of fields are electric and magnetic fields, but simpler systems, such as waves on a string, are also described using fields (e.g. the displacement of a string as a function of position along the string). For now, we'll consider just such a system, where we have a function $\phi(x,t)$ obeying boundary conditions

$$\phi(x=0,t) = \phi(x=L,t) = 0 \tag{1}$$

and satisfying the usual wave equation

$$\rho \frac{\partial^2 \phi}{\partial t^2} - \tau \frac{\partial^2 \phi}{\partial x^2} = 0 \, .$$

Physically, ϕ might represent the displacement of a stretched string (e.g. a guitar string) with two fixed endpoints that is allowed to vibrate in one transverse direction. In this case, ρ and τ would be the mass density and tension of the string. The energy of such a system is given by

$$E = \int_0^L \{ \frac{1}{2}\rho \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2}\tau \left(\frac{\partial \phi}{\partial x} \right)^2 \} dx$$

Our goal is to derive the allowed quantum mechanical energies for such a system.

This looks very different from a single harmonic oscillator, electron in a hydrogen atom, or other simple systems you first considered in quantum mechanics. In particular, the classical system here (i.e. the string) is described by a field rather than a finite set of coordinates (as we have for a system of particles). However, we will now see that the physics is more familiar than you think. The key step is to recall that any function obeying boundary conditions (1) can be written as a sum of sine functions (in musical language, any note can be decomposed into a set of harmonics) i.e. in a Fourier decomposition.

Write an expression showing how $\phi(x,t)$ can be written as a sum of sine functions.

$$\phi(x,t) = \sum_n \phi_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

We now simply want to change variables from the field $\phi(x)$ to the variables ϕ_n which give the amplitude of the nth sine wave in the Fourier decomposition.

Using the expressions above for the wave equation and the energy in the field, write the equation of motion for each ϕ_n and write an expression for the energy of the system in terms of the ϕ_n and their time derivatives. You may find the following useful:

$$\int_0^\pi \sin(mx)\sin(nx) = \frac{\pi}{2}\delta_{m,n} \qquad \int_0^\pi \cos(mx)\cos(nx) = \frac{\pi}{2}\delta_{m,n}\,.$$

We just need to plug in $\phi(x,t) = \sum_n \phi_n(t)\sin\left(\frac{n\pi x}{L}\right)$ into the e.o.m. + the expression for the energy.

E.O.M.: $\rho \frac{\partial^2\phi}{\partial t^2} = \tau \frac{\partial^2 \phi}{\partial x^2}$

$$\Rightarrow \sum_n \frac{\partial^2 \phi_n}{\partial t^2}\sin\left(\frac{n\pi x}{L}\right) = -\frac{\tau}{\rho}\left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\phi_n$$

$$\Rightarrow \sum_n \left(\frac{\partial^2\phi_n}{\partial t^2} + \frac{\tau}{\rho}\left(\frac{n\pi}{L}\right)^2\phi_n\right)\sin\left(\frac{n\pi x}{L}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial^2\phi_n}{\partial t^2} = -\frac{\tau}{\rho}\left(\frac{n\pi}{L}\right)^2\phi_n}$$

Energy:

$$E = \int_0^L dx \frac{1}{2}\rho \sum_n \dot\phi_n \sin\left(\frac{n\pi x}{L}\right)\sum_m \dot\phi_m \sin\left(\frac{m\pi x}{L}\right) + \frac{1}{2}\tau\left(\sum_n \phi_n\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right)\left(\sum_m \phi_m \frac{m\pi}{L}\cos\left(\frac{m\pi x}{L}\right)\right)$$

$$\Rightarrow \boxed{E = \sum_n\left(\frac{L}{4}\rho\dot\phi_n^2 + \frac{L}{4}\tau\phi_n^2\left(\frac{n\pi}{L}\right)^2\right)}$$

Equivalence to a system of harmonic oscillators

One of the simplest systems to study in quantum mechanics is the harmonic oscillator, which classically has equation of motion

$$\ddot{x} = -\omega^2 x$$

For a collection of harmonic oscillators with masses m_n and frequencies ω_n, the total energy is

$$E = \sum_n \{\frac{1}{2}m_n\dot{x}_n^2 + \frac{1}{2}m_n\omega_n^2 x_n^2\} \ .$$

You should now be able to see that in terms of the ϕ_n variables, the equations of motion and the energy for our string are exactly the same as those for a collection of harmonic oscillators.

What masses and freqencies do we need to choose for a collection of harmonic oscillators so that the equations of motion and the classical expression for the energy will be exactly the same as for the string?

From E.o.M.: $\omega_n = \sqrt{\frac{\tau}{\rho}}\frac{n\pi}{L}$

From Energy expression: $m_n = \frac{L\rho}{2}$

Quantum mechanically, what are the allowed energies for such a system of harmonic oscillators, relative to the ground state energy? Can you write a general formula for the allowed energies?

For H.Osc. with freq ω_n: $E = \hbar\omega_n(N_n + \frac{1}{2})$

ground state energy

Total energy relative to ground state:

$$E = \sum \hbar\omega_n \cdot N_n$$

$$\Rightarrow \boxed{E = \sum_n \sqrt{\frac{\tau}{\rho}} \cdot \hbar \cdot \frac{n\pi}{L} \cdot N_n}$$

How do your expressions for the last [redacted] question[redacted] compare with your answer[redacted] from the first page?

exactly the same if we identify $\sqrt{\frac{\tau}{\rho}} \longleftrightarrow c$

velocity of 1D waves

velocity of light.

Hopefully, what you've learned so far is that our field theory is mathematically equivalent at the classical level to a system of harmonic oscillators, and that this system of harmonic oscillators has the same quantum energy spectrum as a system with arbitrary numbers of photons. If you believe the quantum energy spectrum of the field theory is really the same as the quantum energy spectrum of the harmonic oscillators, then we have managed to show that **our simple quantum field theory has the same quantum spectrum as a system with *arbitrary* numbers of particles.**

But is the quantum spectrum of our field theory really the same as the quantum spectrum of the set of harmonic oscillators? The answer must be yes, since the rules for quantizing a classical system depend only on the action (or Lagrangian) for the system, and our two systems with the same equations of motion and energy must also have the same action. Nevertheless, it may be helpful to review the formal procedure:

To quantize the system, we define "momenta" $p_n = M\dot{\phi}_n$[1] and rewrite the energy in terms of ϕ_n and p_n to get the Hamiltonian:

$$H = \sum_n \{\frac{p_n^2}{2M} + \frac{1}{2}M\omega_n^2\phi_n\}$$

In the quantum picture, the "coordinates" ϕ_n and the corresponding momenta p_n are operators obeying the fundamental commutation relation

$$[\phi_n, p_n] = i\hbar \ .$$

As with any system of harmonic oscillators, the energy eigenstates are obtained by acting on a ground state with creation operators

$$a_n^\dagger = \frac{1}{\sqrt{2\hbar M\omega_n}}(-ip_n + m\omega_n\phi_n)$$

where the energy coming from each individual oscillator is

$$E_n = \hbar\omega_n(N_n + \frac{1}{2})$$

[1] The general rule is that we should write down the Lagrangian that gives the desired equation of motion and define the momentum corresponding to any variable q as $p = \frac{\delta L}{\delta q}$.