Transition amplitudes

Consider a quantum field theory with Hamiltonian $H = H_0 + H_I$ where H_0 represents the part quadratic in the fields. The interacting part of the Hamiltonian can lead to transitions which change the number and/or properties of the particles in our state. Here, we would like to derive a convenient formula for the probability amplitudes associated with such transitions.

Even though our theory is interacting, it will still be convenient to use a basis of states inherited from the free Hamiltonian. We'll imagine that we have some eigenstate of the free Hamiltonian H_0 at time $t = t_0$. Then we evolve forward in time and ask for the probability amplitude that at time t we will find some other basis element if we measure the system. More general transition amplitudes can be expressed in terms of these ones involving the basis elements.

Q: To start, write down a basis of energy eigenstates for the free Hamiltonian H_0.

$$|0\rangle, \quad a^\dagger_{\vec{p}}|0\rangle, \quad a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2}|0\rangle \ldots$$

Assume that the states in the previous question are defined at $t = 0$. It will be convenient below to use a basis for the states at other times which is just the previous basis evolved forward to the new time t using the free Hamiltonian H_0.

Q: If $|\Psi(t=0)\rangle$ is one of the basis elements from the previous question, write a formula for the corresponding basis element $|\Psi(t)\rangle$ at time t.

$$|\Psi(t)\rangle = e^{-iH_0 t}|\Psi(t=0)\rangle$$

Q: Now, suppose we have a general state $|\Psi_0\rangle$ at $t = t_0$. What is probability amplitude that if we measure the system at time t, we will find state $|\Psi_1\rangle$ (assuming that this state is an eigenstate corresponding to the possible result of a measurement)?

$$\text{amplitude} = \cancel{\langle \Psi_1 | e^{-iH(t)} | \Psi_0 \rangle}$$

$$= \langle \Psi_1 | e^{-iH(t-t_0)} | \Psi_0 \rangle$$

Using your answers from the previous questions, the transition amplitude from a basis element $|\Psi_1(t_0)\rangle$ at time t_0 to the basis state $|\Psi_2(t)\rangle$ at time t can be written as

$$\langle\Psi_2(0)|U(t,t_0)|\Psi_1(0)\rangle \ .$$

Q: Write a formula for $U(t,t_0)$ in terms of the Hamiltonians H_I and H_0 and the times t and t_0.

Have: $\langle \Psi_2 | e^{-iH(t-t_0)} | \Psi_1 \rangle$

$$= \langle \Psi_2(0) | e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} | \Psi_1(0) \rangle$$

$$\therefore \quad U(t,t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}$$

We now want to write $U(t,t_0)$ in a more useful form. Let's define the time-dependent operator $H_I(t)$ by

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} \ .$$

From the definition, we can see that $H_I(t)$ is obtained from H_I simply by replacing the fields $\phi(x)$ with the time-dependent fields $\phi(x,t)$ we have defined before. To go further, let's see what U looks like for infinitesimal times.

Q: For $t = t_0 + dt$, write a formula for $U(t,t_0)$, expanded to order dt. Express the result in terms of the time-dependent H_I.

$$U(t_0+dt, t_0) = e^{+iH_0(t_0+dt)} e^{-iH dt} e^{-iH_0 t_0}$$

$$= e^{+iH_0 t_0} (1 + iH_0 dt)(1 - iH_0 dt - iH_I dt) e^{-iH_0 t_0}$$

$$= e^{iH_0 t_0} (1 - iH_I dt) e^{-iH_0 t_0}$$

$$= \mathbb{1} - i H_I(t_0)\, dt$$

Q: Reexpress this in terms of an exponential that agrees with your previous result up to order dt^2.

$$U(t_0+dt, t_0) = e^{-iH_I(t_0)dt} + \mathcal{O}(dt^2)$$

Now, the evolution over a finite time can be obtained by breaking up the time interval into many parts of size dt, and writing

$$U(t,t_0) = \lim_{dt\to 0} U(t,t-dt)U(t-dt,t-2dt)\cdots U(t0+dt,t0) \tag{1}$$

Q: Rewrite the right-hand-side of this equation using your exponential expression for $U(t+dt,t)$.

$$= e^{-i H_I[t-dt]dt}\, e^{-i H_I[t-2dt]dt} \cdots e^{-i H_I[t_0]dt}$$

The above expression defines what is known as the time-ordered exponential:

$$U(t,t_0) = T\left\{e^{-i\int_{t_0}^{t} H_I(t)dt}\right\} .$$

In practice, it is much more convenient to have an expression for this expanded order by order in H_I. To obtain this (and to see why the time-ordered exponential is written in this way) start again with (1), but now write it out in terms of the infinitesimal expression $U = 1 + \ldots$ you derived above and **write down all terms in (1) that are linear in H_I. Express the complete set of these in terms of an integral.**

These terms come from taking the linear term in H_I for just one of the exponentials above, and the $(H_I)^0$ term i.e. 1 from the rest:

$$-i H_I[t-dt]dt - i H_I[t-2dt]dt - \ldots - i H_I[t_0]dt$$

$$= -i\int_{t_0}^{t} dt\, H_I[t]$$

Q: Now, in the same way, write down the terms in (1) that are quadratic in H_I. Try to express this set of terms in terms of a double integral. *Hint: be careful about the limits on your integrals, and keep in mind that $H_I(t_1)$ and $H_I(t_2)$ do not commute with each other.*

Now we need one H_I from two different exponentials, we get

$$- \sum_{t_1 > t_2} H_I[t_1] H_I[t_2] \, dt^2 \qquad \left(\text{where } \begin{array}{l} t_1 = t - n \cdot dt \\ t_2 = t - n \cdot dt \end{array}\right)$$

$t \leq t_i \leq t_0$.

$$\underset{dt \to 0}{=} - \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I[t_1] H_I[t_2]$$

This can also be written as

$$-\frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, T \left\{ H_I(t_1) H_I(t_2) \right\}$$

defined to order expressions from larger to smaller times.

Q: Can you figure out an expression for the terms in (1) that are of order n in H_I?

$$(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, H_I(t_1) \cdots H_I(t_n)$$

This is ~~the~~ equivalent to

$$\frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \, T \left\{ H_I(t_1) \cdots H_I(t_n) \right\}$$